

Euclidean analysis of the entropy functional formalism

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ABSTRACT

The attractor mechanism implies that the supersymmetric black hole near horizon solution is defined only in terms of the conserved charges and is therefore independent of asymptotic moduli. Starting only with the *near horizon* geometry, Sen's entropy functional formalism computes the entropy of an extreme black hole by means of a Legendre transformation where the electric fields are defined as conjugated variables to the electric charges. However, traditional Euclidean methods require the knowledge of the *full* geometry to compute the black hole thermodynamic quantities. We establish the connection between the entropy functional formalism and the standard Euclidean formalism taken at zero temperature. We find that Sen's entropy function f (on-shell) matches the zero temperature limit of the Euclidean action. Moreover, Sen's near horizon angular and electric fields agree with the chemical potentials that are defined from the zero-temperature limit of the Euclidean formalism.

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1 Introduction

Black holes (BH) are one of most interesting laboratories we have to investigate quantum gravity effects. Due to their thermodynamic behavior these objects have been associated to ensembles of microstates in the fundamental quantum gravity theory where ideally, quantum statistical analysis should account for all the BH coarse-grained thermodynamical behavior. In particular, many important insights in the classical and quantum structure of BH have been obtained studying supersymmetric configurations in string theory. Supersymmetric BH have many important properties that turn out to be crucial to obtain all the new results. Basically, supersymmetry triggers a number of non-renormalization mechanisms that protect tree level calculations from higher order loop corrections. Moreover, this kind of behavior has also been found in some non-supersymmetric extreme solutions.

1.1 Attractor mechanism and entropy functional formalism

In this context we have the so called attractor mechanism [1]. It was originally thought in the context of four dimensional $N = 2$ supergravity, where we have that the values of the scalar fields at the horizon are given by the values of the BH conserved charges and are independent of the asymptotic values of the scalars at infinity. For these BH (and others) it has been checked that the Bekenstein-Hawking entropy agrees with the microscopic counting of the associated D-brane system. Not only in the supergravity approximation, but also after higher derivative corrections are added to the generalized prepotential [2]. These results motivated a conjecture where the BH partition function equals the squared of the associated topological string partition function *i.e.*, $Z_{BH} = |Z_{Top}|^2$ [3]. Lately, the attractor mechanism has been extended to other directions, and applied to several gauged and ungauged supergravities (see, *e.g.*, [4, 5, 6]).

Importantly, the attractor mechanism has provided a new way to calculate the BH entropy. In a series of articles [7, 8, 9], Sen recovered the entropy of D -dimensional BPS BH using only the near horizon part of the geometry. Basically, in this regime the solution adopts the form $AdS_2 \otimes S^{D-2}$ ¹ plus some electric and magnetic fields. The entropy S is obtained by introducing a function f as the integral of the corresponding supergravity Lagrangian over the S^{D-2} . More concretely, an entropy function is defined as 2π times the Legendre transform of f with respect to the electric fields e_i . Then, an extremization procedure fixes the on-shell BPS values of the different fields of the solution and in particular determines the BPS value of the entropy,

$$S_{bps} = 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f \right)_{bps}. \quad (1.1)$$

Note that in the above definition the different near horizon electric fields take the role of “conjugated chemical potentials” to the BH charges. This formalism has also been extended to extreme non-BPS BH.

The attractor mechanism, both for asymptotically AdS or flat BH, implies that in the near horizon geometry we have a dual CFT theory where the microscopic structure can be studied. We expect that not only the entropy but *all* the statistic properties of such supergravity systems should be described in terms of their dual CFT states.

1.2 Zero temperature limit and chemical potentials

Supersymmetric BH in asymptotically AdS spaces have also been studied using the AdS/CFT correspondence [10, 11, 12, 13]. For the AdS_5 case we still do not have a CFT microscopic derivation of its entropy that reproduces the supergravity result. Nevertheless, in [12, 13] it was showed that the phase space of this supersymmetric sector can be scanned in both sides of the correspondence showing a rich structure with phase transitions and Hagedorn alike behavior². In fact, observables in both dual pictures agree up to numerical factors, a very non-trivial result since the CFT calcu-

¹The analysis of the near horizon geometry has been applied to more general BH that define squashed $AdS_2 \otimes S^{D-2}$ geometries like in [9, 4].

²These $T = 0$ phase transitions were analyzed both in the strong and weak coupling regimes. Remarkably, it was found that their properties resemble the well-known finite temperature phase transitions, where the Hawking-Page phase transition in the strong coupling corresponds to the deconfinement/confinement transition at weak coupling [10, 13].

lations are performed at zero coupling only³. In order to study the full statistical properties (so that we could in principle do more than just account for the entropy), in [12, 13] it was found how to define the different chemical potentials μ_i that control the supersymmetric BH partition function in the grand canonical ensemble. The basic input comes from the thermodynamics of the dual CFT theory, where the BPS partition function is obtained from the finite temperature one, by sending the temperature to zero. This also sends the several chemical potentials to their BPS values. The associated dual limiting procedure in the supergravity regime corresponds also to send the temperature to zero. Done carefully, this defines the supergravity chemical potentials that are dual to the the CFT ones and, more generally, the statistical mechanics of *supersymmetric* BH that is free of divergencies. These chemical potentials are the next to leading order terms of the zero temperature expansion of the horizon angular velocities and electric potentials. The resulting supergravity partition function is given, as expected, by the exponentiation of the regularized Euclidean action I evaluated at the BH solution. In this paper we call “Euclidean zero-temperature formalism” to the zero-temperature limit in the supergravity system that determines the Euclidean action, entropy and the chemical potentials. After some algebra we arrive to the supersymmetric quantum statistical relation (SQSR) [14] where the Euclidean action I can be rewritten as the Legendre transform of the entropy S with respect to the different supersymmetric chemical potentials μ_i ,

$$I_{bps} = \mu_i q_{bps}^i - S_{bps} , \quad (1.2)$$

where q_{bps}^i ’s represent the conserved BH charges conjugated to the μ_i ’s (later, we will use the notation $q^i \equiv \{Q^i, J^i\}$ and $\mu^i \equiv \{\phi^i, \omega^i\}$). As said above, these supergravity chemical potentials are closely related to the dual CFT chemical potentials. Therefore, they provide a very clear picture of the BPS BH as dual to a supersymmetric CFT in the grand canonical ensemble. This approach also defines the finite supersymmetric Euclidean action (1.2), and in fact allows to study the statistical mechanics of BPS black holes. A similar analysis can be done for extreme non-BPS systems.

1.3 Entropy functional formalism from an Euclidean perspective

Sen’s entropy functional formalism is formulated only with the knowledge of the near horizon geometry. But, since it computes the BH entropy, which is a thermodynamic quantity, it should be possible to understand it starting from a traditional thermodynamical Euclidean analysis of the black hole system.

In fact, the strong resemblance between equations (1.1) and (1.2) is evident. In other words, it would be strange if *string theory produces two unrelated functions in the same supergravity regime that calculate the BH entropy*. Looking into both definitions with more care, we find that the entropy is defined as the Legendre transform of the BH charges in the saddle point approximation of the supergravity theory. Nevertheless, in (1.1) the vacuum solution is just the near horizon geometry with conjugated potentials related to the electric fields, and f is the on-shell Lagrangian over only S^{D-2} . Instead, in (1.2), the vacuum is the entire BH solution; the conjugated potentials are associated to gauge potentials rather than field strengths; and I is the on-shell full Euclidean action. The main goal of this paper is to understand the connection between these two approaches.

³In [10] the CFT partition function was calculated at zero coupling. Also, an index was considered to count supersymmetric states but unfortunately it turns out to be blind to the BH sector.

One of the key points of our analysis relies in the natural splitting of the Euclidean action into two parts corresponding basically to: i) the near horizon part of space, and ii) the asymptotic region. Then we find that in the extremal cases (without ergoregion), the asymptotic part vanishes, and the near horizon part reduces to Sen's function $2\pi f$. Also, the conjugated chemical potentials found in both methods agree, due to an argument that relates differences of gauge potentials produced by variations of near-BPS parameters with variations of the potential on the radial coordinate.

1.4 Main results and structure of the paper

As stated above, the main goal of this article is to provide a bridge between Sen's entropy functional formalism and standard Euclidean analysis of the thermodynamics of a black hole system. While doing so, we also find that the supergravity conjugated potentials defined in Sen's formalism map into chemical potentials of the dual CFT.

We obtain a unifying picture where:

1) We are able to recover the entropy function of Sen from the zero temperature limit of the usual BH thermodynamics and the statistical mechanics definitions of the dual CFT theory. The supergravity and their dual CFT chemical potentials are identified with the surviving Sen's near horizon electric and angular fields. The Euclidean action is identified with Sen's function $2\pi f$.

2) As a byproduct of the above analysis we have understood how to calculate the BPS chemical potentials that control the statistical properties of the BH using only the BPS regime, *i.e.*, without needing the knowledge of the non-BPS geometry. The CFT chemical potentials are dual to the supergravity ones. Traditionally, to compute the latter we have to start with the non-BPS solution and send the temperature to zero to find the next to leading order terms in the horizon angular velocities and electric potentials expansions that give the chemical potentials. This requires the knowledge of the non-BPS geometry. Unfortunately, sometimes this is not available and we only know the BPS solution. But, from item 1) we know that the near horizon fields, that Sen computes with the single knowledge of the BPS near horizon solution, give us the supergravity chemical potentials. So now we can compute the supergravity chemical potentials of any BPS BH solution, regardless of its embedding into a family of non-BPS solutions, while still keeping the relation with the dual CFT.

3) It is known that the attractor mechanism seems to work also for non-supersymmetric but extremal BH⁴. We have tested the Euclidean zero temperature formalism for many of these BH, always finding a well defined limit and agreement with Sen's results for extremal non-BPS BH⁵. This is a non-trivial fact since there is no supersymmetry protecting the limit. Therefore, in general, the supergravity regime should not give the correct statistical relations. We interpret this result as another confirmation that there is a protecting mechanism for extremal non-supersymmetric BH.

The plan of the paper is the following. In section 2 we review Sen's entropy functional approach using the D1-D5-P system as an illuminating example. In the beginning of section 3 we review the main ideas and results of the Euclidean zero temperature formalism for BH in the AdS/CFT

⁴See [9, 25] and references there in.

⁵Actually at the level of two derivative theory, Euclidean $T = 0$ formalism is well defined only for BH with no ergoregion. For BH with ergoregion we have an ill-defined limit, that nevertheless allows to define the entropy and all chemical potentials. This is telling us that these geometries are not fully protected from string corrections. The same caveats and conclusions are also obtained using Sen's approach, and this is related to the fact that for these BH the attractor mechanism is only partial since there is dependance on the asymptotic data [9].

framework. Then, we apply this formalism to the most general rotating D1-D5-P system. We analyze the connection between the entropy functional and Euclidean formalisms in section 4, identifying how and why both prescriptions are equivalent. In section 5 we discuss the application of the Euclidean $T = 0$ formalism to extreme non-BPS BH and again find agreement with Sen's results. Section 6 is devoted to a short discussion on the results and possible future avenues to follow. In Appendix A we review the D1-D5-P BH solution in detail, including its thermodynamics. In Appendix B, we write the chemical potentials and Euclidean action for some other BH systems not considered in the main body of the text. We consider the four charged system of type IIA supergravity, and the Kerr-Newman BH. We confirm that for these BH the relation established in section 4 between the entropy functional and Euclidean formalisms holds. This agreement also extends to AdS black holes as is explicitly confirmed in the context of 5D gauged supergravity in [15].

Note: While we were proof-reading this article, the paper [16] appeared in the arXives. It contains relevant discussions and results connected to our work, regarding Sen's approach and Wald's method for AdS BH.

2 Entropy functional formalism revisited

As we pointed out in the introduction, Sen developed a simple method – the entropy functional formalism – to compute the entropy of supersymmetric BH in supergravity [7]. Lately, this approach has been applied to rotating BH in gauged and ungauged supergravity (see, *e.g.*, [9, 6]). Here, we will review some of the key aspects of this formalism that we will use latter. We just need to address non-rotating cases, but we will comeback to rotating attractors at the end of this section, for completeness.

Sen's entropy functional formalism assumes that: (i) we start with a Lagrangian \mathcal{L} with gravity plus some field strengths and uncharge massless scalar fields; and (ii) due to the attractor mechanism the near horizon geometry of a D -dimensional BH is set to be of the form $AdS_2 \otimes S^{D-2}$. From the above input data, the general form of the near horizon BH solution is

$$\begin{aligned} ds^2 &= v_1 \left(-\rho^2 d\tau + \frac{d\rho^2}{\rho^2} \right) + v_2 d\Omega_{D-2}^2, \\ F_{\rho\tau}^{(i)} &= e_i, \quad H^{(a)} = p_a \epsilon_{D-2}, \\ \phi_s &= u_s, \end{aligned} \tag{2.1}$$

where ϵ_{D-2} is the unit-volume form of S^{D-2} , and (e_i, p_a) are respectively the electric fields and the magnetic charges of the BH. Note that $(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ are arbitrary constants up to now and therefore the solution is off-shell. Next, it is defined the following function

$$f(\vec{u}, \vec{v}, \vec{e}, \vec{p}) = \int_{S^{D-2}} \sqrt{-g} \mathcal{L}, \tag{2.2}$$

where \mathcal{L} is the string frame Lagrangian of the theory (see, *e.g.*, (A.19)). After minimizing $f(\vec{u}, \vec{v}, \vec{e}, \vec{p})$ with respect to (\vec{u}, \vec{v}) we obtain the exact supersymmetric near horizon BH solution in terms of (\vec{e}, \vec{p}) . In fact, the field equations are reproduced by this minimization procedure. Furthermore, minimization with respect to \vec{e} gives the electric charges \vec{q} . Explicitly, the on-shell values of $\vec{u}, \vec{v}, \vec{e}$

that specify (2.1) for a given theory described by (2.2) are found through the relations,

$$\frac{\partial f}{\partial u_s} = 0, \quad \frac{\partial f}{\partial v_j} = 0, \quad \frac{\partial f}{\partial e_i} = q_i. \quad (2.3)$$

Then, using Wald formalism [27], Sen derived that the entropy S of the corresponding BH is given by 2π times the Legendre transform of f ,

$$S = 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f \right). \quad (2.4)$$

Finally notice that the minimization procedure, can be taken only after S is defined. In this form S is really an entropy function of $(\vec{u}, \vec{v}, \vec{q}, \vec{p})$, that after minimization equals the BH entropy as a function of (\vec{q}, \vec{p}) only.

In the rest of this section we will discuss the above formalism in a specific theory. We consider the D1-D5-P supersymmetric solution of ten-dimensional type IIB supergravity, discussed in the previous section, as the main example (this case was first analyzed in [17], at the level of supergravity and for its higher order corrections). Our aim is to highlight the details of the application of Sen's formalism to this solution. This will provide a solid background to compare, in section 4, Sen's formalism with the Euclidean one developed in section 3.

From Appendix A.1 we know that the supersymmetric D1-D5-P metric, the RR two-form $C_{(2)}$ and the dilaton Ψ are given by ⁶

$$ds^2 = \frac{1}{\sqrt{H_1 H_5}} [-dt^2 + dy^2 + \frac{Q_p^{bps}}{r^2} (dt - dy)^2] + \sqrt{H_1 H_5} (dr^2 + r^2 d\Omega_3^2) + \sqrt{H_1/H_5} \sum_{i=1}^4 dz_i^2, \\ C_{(2)} = -\frac{Q_1^{bps}}{r^2 H_1} dt \wedge dy - Q_5^{bps} \cos^2 \theta d\phi \wedge d\psi, \quad e^{2\Psi} = \frac{H_1}{H_5}. \quad (2.5)$$

where $H_1 = (1 + \frac{Q_1^{bps}}{r^2})$, $H_5 = (1 + \frac{Q_5^{bps}}{r^2})$ and $(Q_1^{bps}, Q_5^{bps}, Q_p^{bps})$ are the D1, D5, P charges, respectively. Then, it is easy to take the near horizon limit to obtain,

$$ds^2 = \frac{\sqrt{Q_1^{bps} Q_5^{bps}}}{4} \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + \sqrt{Q_1^{bps} Q_5^{bps}} d\Omega_3^2 \\ + \frac{Q_p^{bps}}{\sqrt{Q_1^{bps} Q_5^{bps}}} \left(dz + \frac{\sqrt{Q_1^{bps} Q_5^{bps}}}{2\sqrt{Q_p^{bps}}} \rho d\tau \right)^2 + \sqrt{Q_1^{bps}/Q_5^{bps}} \sum_{i=1}^4 dz_i^2, \\ F_{(3)} = \frac{1}{2} \sqrt{\frac{Q_5^{bps} Q_p^{bps}}{Q_1^{bps}}} d\rho \wedge d\tau \wedge dz + 2Q_5^{bps} \epsilon_3, \quad e^{2\Psi} = \frac{Q_1^{bps}}{Q_5^{bps}}, \quad (2.6)$$

where we used

$$\tau = \frac{2}{\sqrt{Q_1^{bps} Q_5^{bps} Q_p^{bps}}} t, \quad \rho = r^2, \quad z = y - t. \quad (2.7)$$

⁶This is the string frame version of (A.4), and (A.7) and (A.8) with $a_1 = a_2 = 0$.

Note that, alternatively, all the information encoded in the near horizon structure (2.6) could be extracted without knowing the full geometry, using Sen's approach. Its application starts by assuming that the near horizon metric is given in terms of the unknowns $(\vec{v}, \vec{u}, \vec{e}, \vec{p})$ as follows,

$$ds^2 = v_1 \left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2} \right) + v_2 d\Omega_3^2 + u_1 (dz + e_2 \rho dt)^2 + u_2 \sum_{i=1}^4 dz_i^2, \\ F_{(3)} = e_1 d\rho \wedge d\tau \wedge dz + 2Q_5^{bps} \epsilon_3, \quad e^{2\Phi} = u_3^2. \quad (2.8)$$

Having the Lagrangian (A.19) of type IIB at hand, one now follows the steps summarized in (2.2)-(2.4) to find the on-shell expressions for $(\vec{v}, \vec{u}, \vec{e})$. From (2.8) one has $\sqrt{-\tilde{g}} = u_1^{1/2} u_2^2 v_1 v_2^{3/2} \sin \theta \cos \theta$, $F_{\rho\tau z}^{(3)} = e_1$ and $F_{\theta\psi\phi}^{(3)} = 2Q_5^{bps} \sin \theta \cos \theta$. The entropy function, $S(\vec{u}, \vec{v}, \vec{q}, \vec{p}) = 2\pi[q_1 e_1 + q_2 e_2 - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})]$ is then

$$S(\vec{u}, \vec{v}, \vec{q}, \vec{p}) = 2\pi \left\{ q_1 e_1 + q_2 e_2 - \frac{1}{4} u_1^{1/2} u_2^2 v_1 v_2^{3/2} \left[u_3^2 \left(\frac{u_1 e_2^2}{v_1^2} + \frac{12}{v_2} - \frac{4}{v_1} \right) + \frac{e_1^2}{u_1 v_1^2} - \frac{4p^2}{v_2^3} \right] \right\}.$$

Minimizing this entropy function with respect to $\vec{u}, \vec{v}, \vec{e}$ one finds the on-shell attractor values,

$$\vec{v} = \left(\frac{1}{4} \sqrt{Q_1^{bps} Q_5^{bps}}, \sqrt{Q_1^{bps} Q_5^{bps}} \right), \quad \vec{u} = \left(\frac{Q_p^{bps}}{\sqrt{Q_1^{bps} Q_5^{bps}}}, \sqrt{\frac{Q_1^{bps}}{Q_5^{bps}}}, \sqrt{\frac{Q_5^{bps}}{Q_1^{bps}}} \right), \\ \vec{e} = \left(\frac{1}{2} \sqrt{\frac{Q_5^{bps} Q_p^{bps}}{Q_1^{bps}}}, \frac{1}{2} \sqrt{\frac{Q_1^{bps} Q_5^{bps}}{Q_p^{bps}}} \right), \quad \vec{q} = (Q_1^{bps}, Q_p^{bps}), \quad p = Q_5^{bps},$$

One also finds that $f(\vec{q}, \vec{p}) = 0$ on-shell. Plugging this information into the entropy function $S(\vec{u}, \vec{v}, \vec{q}, \vec{p})$ we get

$$S(\vec{q}, \vec{p}) = 2\pi [q_1 e_1 + q_2 e_2 - f]_{on-shell} \\ = 2\pi \sqrt{Q_1^{bps} Q_5^{bps} Q_p^{bps}}, \quad (2.9)$$

that is the well known result for this BH.

It will be relevant for section 4 to stress that the above analysis can be carried on in the case where the magnetic field is replaced by its dual electric field. This electric field comes from the RR seven-form field strength $F_{(7)}$, Poincaré dual of the magnetic part of $F_{(3)}$,

$$F_{(7)} = \frac{2Q_5^{bps}}{r^3 H_5^2} dr \wedge dt \wedge dy \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4. \quad (2.10)$$

In the near horizon limit, *i.e.*, after taking the change of coordinates (2.7), $F_{(7)}$ reduces to

$$F_{(7)} = \frac{1}{2} \sqrt{\frac{Q_1^{bps} Q_p^{bps}}{Q_5^{bps}}} d\rho \wedge d\tau \wedge dz \wedge dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4. \quad (2.11)$$

In the next lines we want to recover this near horizon attractor value for $F_{(7)}$, without making use of the near horizon limit of the full geometry, *i.e.*, using instead a Sen-like approach.

In this pure electric case, we first notice that there is an extra pair of conjugated variables (e_3, q_3) and second, that f should be now calculated on a modified Lagrangian with the $F_{(7)}$ RR field strength appropriately added. This is an effective “democratic” Lagrangian supplemented by duality constraints imposed by hand⁷. The motivation, limitations and formulation of this effective Lagrangian are presented in detail in [18]. In this context, the string frame Lagrangian (A.19) of the D1-D5-P system takes the form,

$$\mathcal{L} = \frac{1}{16\pi G_{10}} \left[e^{-2\Psi} \left(\tilde{R} - 4\partial_\mu \Psi \partial^\mu \Psi \right) - \frac{1}{2 \cdot 3!} F_{(3)}^2 - \frac{1}{2 \cdot 7!} F_{(7)}^2 \right], \quad (2.12)$$

where the *magnetic* part of the original $F_{(3)}$ field is now encoded in the $F_{(7)}$ contribution. The D1-branes and D5-branes source the electric $F_{(3)}$ and $F_{(7)}$ fields, respectively.

In the entropy function formalism, the function $f(\vec{u}, \vec{v}, \vec{e})$ is obtained by evaluating action (2.12) at the horizon, *i.e.*, by integrating along the S^8 sphere. We use the near-horizon fields (2.8). So, the metric determinant is $\sqrt{-\tilde{g}} = u_1^{1/2} u_2^2 v_1 v_2^{3/2} \sin \theta \cos \theta$, $F_{\rho\tau z}^{(3)} = e_1$ and $F_{\rho\tau z z_1 \dots z_4}^{(7)} = e_3$. The entropy function, $S(\vec{u}, \vec{v}, \vec{q}) = 2\pi[q_1 e_1 + q_2 e_2 + q_3 e_3 - f(\vec{u}, \vec{v}, \vec{e})]$ is then

$$S(\vec{u}, \vec{v}, \vec{q}) = 2\pi \left\{ q_1 e_1 + q_2 e_2 + q_3 e_3 - \frac{1}{4} u_1^{1/2} u_2^2 v_1 v_2^{3/2} \left[u_3^2 \left(\frac{u_1 e_2^2}{v_1^2} + \frac{12}{v_2} - \frac{4}{v_1} \right) + \frac{e_1^2}{u_1 v_1^2} + \frac{e_3^2}{u_1 v_1^2 u_2^4} \right] \right\}.$$

Minimizing this entropy function with respect to $\vec{u}, \vec{v}, \vec{e}$ one finds the on-shell attractor values

$$\begin{aligned} \vec{v} &= \left(\frac{1}{4} \sqrt{Q_1^{bps} Q_5^{bps}}, \sqrt{Q_1^{bps} Q_5^{bps}} \right), & \vec{u} &= \left(\frac{Q_p^{bps}}{\sqrt{Q_1^{bps} Q_5^{bps}}}, \sqrt{\frac{Q_1^{bps}}{Q_5^{bps}}}, \sqrt{\frac{Q_5^{bps}}{Q_1^{bps}}} \right), \\ \vec{e} &= \left(\frac{1}{2} \sqrt{\frac{Q_5^{bps} Q_p^{bps}}{Q_1^{bps}}}, \frac{1}{2} \sqrt{\frac{Q_1^{bps} Q_5^{bps}}{Q_p^{bps}}}, \frac{1}{2} \sqrt{\frac{Q_1^{bps} Q_p^{bps}}{Q_5^{bps}}} \right), & \vec{q} &= (Q_1^{bps}, Q_p^{bps}, Q_5^{bps}), \end{aligned} \quad (2.13)$$

which are used to obtain the on-shell function: $f(\vec{q}) = \frac{1}{2} \sqrt{Q_1^{bps} Q_5^{bps} Q_p^{bps}}$. Then, use of equation (2.13) yields the on-shell entropy value,

$$\begin{aligned} S(\vec{q}) &= 2\pi [q_1 e_1 + q_2 e_2 + q_3 e_3 - f]_{on-shell} \\ &= 2\pi \sqrt{Q_1^{bps} Q_5^{bps} Q_p^{bps}}, \end{aligned} \quad (2.14)$$

that is, in this dual computation we indeed recover the value (2.9).

As commented in the introduction, the above approach was generalized to rotating BH in ungauged and gauged supergravities [9, 6]. At the level of two derivative Lagrangian, rotating BH in ungauged supergravity have their near horizon geometry fully determined by the entropy functional only if they have no ergoregion. However, BH with ergoregion show only partial attractor mechanism, since their entropy functional has flat directions [9, 26]. In this case, minimization does not fix the value of all quantities in the near horizon geometry. There is some surviving dependance

⁷We should emphasize that the introduction of a RR p -form field strength with $p > 5$ doubles the number of degrees of freedom. To get the right equations of motion from (2.12) we must then introduce by hand duality constraints relating the lower- and higher-rank RR potentials. We ask the reader to see [18] for further details.

on the asymptotic value of the scalars, although it fixes the form of entropy itself and the electric and angular fields.

Generalization to gauged supergravities includes AdS BH into the discussion. The resulting picture is basically the same, where care has to be taken when evaluating f due to Chern-Simon terms in the Lagrangian (see [6] for details). In these cases, the attractor mechanism is related to a non trivial flow between fixed points at both boundaries of spacetime, the horizon AdS and the asymptotic AdS at infinity.

3 Euclidean zero-temperature formalism: BPS black holes

In [12, 13] the “thermodynamics” or better “the statistical mechanics” of supersymmetric solitons in gauged supergravity was studied in detail using an extension of standard Euclidean thermodynamical methods to zero temperature systems. We call this approach the Euclidean zero-temperature formalism. BPS BH can be studied as dual configurations of supersymmetric ensembles at zero temperature *but* non-zero chemical potentials in the dual CFT. These potentials control the expectation value of the conjugated conserved charges carried by the BH, like *e.g.*, angular momenta and electric charge.

In these articles, the two main ideas are: **First**, there is a supersymmetric field theory dual to the supergravity theory. **Second**, in this dual field theory the grand canonical partition function over a given supersymmetric sector can be obtained as the zero temperature limit of the general grand canonical partition function at finite temperature. This limit also fixes the values of several chemical potentials of the system.

To make things more clear, recall that all supersymmetric states in a field theory saturate a BPS inequality that translates into a series of constraints between the different physical charges. For definiteness, let us consider a simple case where the BPS bound corresponds to the constraint⁸: $E = J$. Then, defining the left and right variables $E^\pm = \frac{1}{2}(E_\nu \pm J_\nu)$, $\beta_\pm = \beta(1 \pm \Omega)$ the grand canonical partition function is given by

$$Z_{(\beta, \Omega)} = \sum_{\nu} e^{-(\beta_+ E_+ + \beta_- E_-)} . \quad (3.1)$$

At this point, it is clear that taking the limit $\beta_- \rightarrow \infty$ while $\beta_+ \rightarrow \omega$ (constant), gives the correct supersymmetric partition function. The above limiting procedure takes T to zero, *but* also scales Ω in such a way that the new supersymmetric conjugated variable ω is finite and arbitrary. Note that among all available states, only those that satisfy the BPS bound are not suppressed in the sum, resulting in the supersymmetric partition function

$$Z(\omega) = \sum_{bps} e^{-\omega J} , \quad (3.2)$$

where the sum is over all supersymmetric states (*bps*) with $E = J$. The above manipulations are easy to implement in more complicated supersymmetric field theories like, *e.g.*, $N = 4$ SYM theory in four dimensions. What is less trivial is that amazingly it could also be implemented in the

⁸This type of BPS bound appears in two dimensional supersymmetric models like, *e.g.*, the effective theory of 1/2 BPS chiral primaries of $N = 4$ SYM in $R \otimes S^3$ (see [19, 20, 21]).

dual supersymmetric configurations of gauged supergravity, since it means that these extreme BPS solutions are somehow protected from higher string theory corrections.

Before we apply the Euclidean zero-temperature formalism to concrete black hole systems, it is profitable to highlight its key steps. To study the statistical mechanics of supersymmetric black holes we take the off-BPS BH solution and we send $T \rightarrow 0$. In this limiting procedure, the angular velocities and electric potentials at the horizon can be written as an expansion in powers of the temperature. More concretely one has when $T \rightarrow 0$,

$$\begin{aligned}\beta &\rightarrow \infty, \\ \Omega &\rightarrow \Omega_{bps} - \frac{\omega}{\beta} + \mathcal{O}(\beta^{-2}), \\ \Phi &\rightarrow \Phi_{bps} - \frac{\phi}{\beta} + \mathcal{O}(\beta^{-2}),\end{aligned}\tag{3.3}$$

where β is the inverse temperature; (Ω, Φ) are the angular velocities and electric potentials at the horizon; the subscript *bps* stands for the values of these quantities in the on-shell BPS solution; and (ω, ϕ) are what we call the supersymmetric conjugated potentials, *i.e.*, the next to leading order terms in the expansion. For all the systems studied, we find that the charges have the off-BPS expansion,

$$E = E^{bps} + \mathcal{O}(\beta^{-2}), \quad Q = Q^{bps} + \mathcal{O}(\beta^{-2}), \quad J = J_{\phi}^{bps} + \mathcal{O}(\beta^{-2}),\tag{3.4}$$

where (E, Q, J) are the energy, charges and angular momenta of the BH. In supergravity, the grand canonical partition function in the saddle point approximation is related to so called quantum statistical relation (QSR) [14]

$$I_{(\beta, \Phi, \Omega)} = \beta E - \Phi Q - \Omega J - S,\tag{3.5}$$

where S is the entropy, and (β, Φ, Ω) are interpreted as conjugated potentials to E, Q, J , respectively. I is the Euclidean action (evaluated on the off-BPS BH solution) that, in this ensemble, depends only on (β, Φ, Ω) . It plays the role of free energy divided by the temperature. Inserting (3.3) and (3.4) into (3.5) yields

$$I_{(\beta, \Phi, \Omega)} = \beta(E^{bps} - \Phi_{bps}Q^{bps} - \Omega_{bps}J^{bps}) + \phi Q^{bps} + \omega J^{bps} - S_{bps} + \mathcal{O}(\beta^{-1}).\tag{3.6}$$

Here, we observe that this action is still being evaluated *off*-BPS. Moreover, the term multiplying β boils down to the BPS relation between the charges of the system and thus vanishes (this will become explicitly clear in the several examples we will consider). This is an important feature, since now we can finally take the $\beta \rightarrow \infty$ limit yielding relation (1.2). With the present notation it reads as

$$I_{bps} = \phi Q^{bps} + \omega J^{bps} - S_{bps}.\tag{3.7}$$

It is important to stress that this zero temperature limiting procedure yields a finite, not diverging, supersymmetric version of QSR, or shortly SQSR. Note that if we had evaluated the Euclidean action (3.5) directly on-shell it would not be well defined, as is well-known. As a concrete realization, we picked (and will do so along the paper) the SQSR to exemplify that the $T \rightarrow 0$ limit yields well-behaved supersymmetric relations. The reason being that this SQSR relation is the one that will provide direct contact with Sen's entropy functional formalism, which is the main aim of our

study. However, it also provides a suitable framework that extends to the study of the *full* statistical mechanics of supersymmetric black holes.

Euclidean action and chemical potentials of BPS D1-D5-P black holes

As we pointed out in the introduction, due to the attractor mechanism, BH in ungauged supergravity have a dual CFT theory defined in the boundary of its near horizon geometry. Therefore, and in a similar way as for asymptotic AdS spacetimes, these BH should be related to statistical ensembles in the dual CFT. As a direct consequence of this duality, we conclude that in the ungauged case there should also exist a well defined zero temperature limit in the supergravity description that yields the dual CFT chemical potentials.

In what follows, we apply the Euclidean $T \rightarrow 0$ limit to the illuminating example of five-dimensional three charged BH with two angular momenta that can be described as the D1-D5-P system of type IIB supergravity⁹. This solution can also be embedded as a solution of eleven-dimensional supergravity, or as a solution of type IIA, where all these different descriptions are related by dimensional reduction and U -dualities. A detailed review of the D1-D5-P BH solution [22, 23] and its thermodynamic properties needed for our discussion can be found in Appendix A.

In type IIB, the ten-dimensional system can be compactified to five dimensions on $T^4 \times S^1$ with the D5-branes wrapping the full internal space and the D1-branes and KK-momentum on the distinguished S^1 . The length of S^1 is $2\pi R$ and the volume of T^4 is V . We will work in units such that the five-dimensional Newton constant is $G_5 = G_{10}/2\pi RV = \pi/4$. The ten-dimensional solution is characterized by six parameters: a mass parameter, M ; spin parameters in two orthogonal planes, (a_1, a_2) ; and three boost parameters, $(\delta_1, \delta_5, \delta_p)$, which fix the D1-brane, D5-brane and KK-momentum charges. The physical range of M is $M \geq 0$. We assume without loss of generality that $\delta_i \geq 0$ ($i = 1, 5, p$), and $a_1 \geq a_2 \geq 0$ (The solutions with $a_1 a_2 \leq 0$ are equivalent to the $a_1 a_2 \geq 0$ ones due to the symmetries of the solution). We will use the notation $c_i \equiv \cosh \delta_i$, $s_i \equiv \sinh \delta_i$.

The BH charges are: ADM mass E , the angular momenta (J_ϕ, J_ψ) and the gauge charges (Q_1, Q_5, Q_p) associated with the D1-branes, D5-branes and KK momentum. In terms of the parameters describing the solution they are given by

$$\begin{aligned} E &= \frac{M}{2} [\cosh(2\delta_1) + \cosh(2\delta_5) + \cosh(2\delta_p)], \\ J_\phi &= -M(a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p), \\ J_\psi &= -M(a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p), \\ Q_i &= M s_i c_i, \quad i = 1, 5, p. \end{aligned} \tag{3.8}$$

Note that these quantities are invariant under interchange of the δ_i 's. This reflects the equivalence of the several geometries obtained by U -dualities, that also interchange the several gauge charges.

Regarding the thermodynamical properties of these BH, it is convenient for future use to define the left and right temperatures, T_L and T_R , through the relation $\beta = \frac{1}{2}(\beta_L + \beta_R)$ ($\beta = 1/T$ and

⁹We present this case as a main example, but include many others in the Appendix B.

$\beta_{L,R} = 1/T_{L,R}$). Then, using this relation together with (A.6) on (A.17) yields¹⁰

$$\beta_L = \frac{2\pi M (c_1 c_5 c_p - s_1 s_5 s_p)}{[M - (a_2 - a_1)^2]^{1/2}}, \quad \beta_R = \frac{2\pi M (c_1 c_5 c_p + s_1 s_5 s_p)}{[M - (a_2 + a_1)^2]^{1/2}}. \quad (3.9)$$

The BH angular velocities $\Omega^{\phi,\psi}$ and electric potentials $\Phi^{(i)}$ are computed in Appendix A. Here, using (A.6), we rewrite them in terms of the parameters $(M, \delta_1, \delta_5, \delta_p, a_1, a_2)$

$$\Omega^{\phi,\psi} = -\frac{\pi}{\beta} \left[\pm \frac{a_2 - a_1}{[M - (a_2 - a_1)^2]^{1/2}} + \frac{a_2 + a_1}{[M - (a_2 + a_1)^2]^{1/2}} \right], \quad (3.10)$$

$$\Phi^{(i)} = \frac{\pi M}{\beta} \left[\frac{(\tanh \delta_i) c_1 c_5 c_p - (\coth \delta_i) s_1 s_5 s_p}{[M - (a_2 - a_1)^2]^{1/2}} + \frac{(\tanh \delta_i) c_1 c_5 c_p + (\coth \delta_i) s_1 s_5 s_p}{[M - (a_2 + a_1)^2]^{1/2}} \right], \quad (3.11)$$

while the expression for the entropy is

$$S = \pi M \left[\frac{c_1 c_5 c_p + s_1 s_5 s_p}{[M - (a_2 - a_1)^2]^{-1/2}} + \frac{c_1 c_5 c_p - s_1 s_5 s_p}{[M - (a_2 + a_1)^2]^{-1/2}} \right]. \quad (3.12)$$

The BPS limit of the three charged BH is obtained by taking $M \rightarrow 0$, $\delta_i \rightarrow \infty$, $J_\phi + J_\psi \rightarrow 0$ while keeping Q_i fixed. In this supersymmetric regime, the charges satisfy the BPS constraints

$$E^{bps} = Q_1^{bps} + Q_5^{bps} + Q_p^{bps}, \quad J_\psi^{bps} = -J_\phi^{bps}. \quad (3.13)$$

As a first step to define the Euclidean $T \rightarrow 0$ limit, we consider the near-BPS limit of this solution,

$$J_\phi + J_\psi \rightarrow 0; \quad M \rightarrow 0, \quad \delta_{1,5} \rightarrow \infty, \quad Q_{1,5} \text{ fixed}; \quad \delta_p \text{ finite}. \quad (3.14)$$

That is, in the near-BPS limit we keep δ_p large *but* finite. This limit is also often called the dilute gas regime since we are neglecting the interactions between left and right movers. Note that since the three charges can be interchanged by U -dualities, it does not matter which one of the boosts we keep finite. Given this equivalence we choose to keep δ_p finite, without any loss of generality.

Now, to take the $T \rightarrow 0$ limit, we define the off-BPS parameter ε , that measures energy above extremality, to be such that $E \equiv E^{bps} + \varepsilon$. In terms of the solution parameters it is given by $\varepsilon = M e^{-2\delta_p}/4$. The details of the off-BPS expansion that we carry on in the sequel can be found in Appendix A.2. Here we just quote the relevant results. We can expand the left and right temperatures in terms of the off-BPS parameter ε yielding,

$$\beta_L = \frac{\pi Q_1^{bps} Q_5^{bps}}{\sqrt{Q_1^{bps} Q_5^{bps} Q_p^{bps} - (J_\phi^{bps})^2}}, \quad \beta_R = \pi \sqrt{Q_1^{bps} Q_5^{bps}} \frac{1}{\sqrt{\varepsilon}}. \quad (3.15)$$

So the BPS limit corresponds to send the temperature $T \rightarrow 0$ by sending $\beta_R \rightarrow \infty$ while keeping β_L finite (we are left with only left-movers). Hence, we find more appropriate to use β_R as the off-BPS parameter instead of ε . These two quantities are related by the second relation of (3.15).

¹⁰Expressions (3.9)-(3.11) agree with the ones first computed in [24] upon the notation identification $a_1 \rightarrow -l_2$ and $a_2 \rightarrow -l_1$.

We can now expand all the thermodynamic quantities in terms of this off-BPS quantity β_R^{-1} . For the angular velocities and electric potentials, the expansion yields

$$\begin{aligned}\Omega^{\phi,\psi} &= \Omega_{bps}^{\phi,\psi} - \frac{2}{\beta_R} \frac{\mp \pi J_\phi^{bps}}{\left[Q_1^{bps} Q_5^{bps} Q_p^{bps} - (J_\phi^{bps})^2\right]^{1/2}} + \mathcal{O}(\beta_R^{-2}), \\ \Phi^{(i)} &= \Phi_{bps}^{(i)} - \frac{2}{\beta_R} \frac{\pi Q_1^{bps} Q_5^{bps} Q_p^{bps}}{Q_i^{bps} \left[Q_1^{bps} Q_5^{bps} Q_p^{bps} - (J_\phi^{bps})^2\right]^{1/2}} + \mathcal{O}(\beta_R^{-2}).\end{aligned}\quad (3.16)$$

where the BPS angular velocities and electric potentials are

$$\Omega_{bps}^{\phi,\psi} = 0; \quad \Phi_{bps}^{(i)} = 1. \quad (3.17)$$

The expansion of the conserved charges yields

$$\begin{aligned}E &= E^{bps} + \mathcal{O}(\beta_R^{-2}), \quad J_\phi = J_\phi^{bps} + \mathcal{O}(\beta_R^{-2}), \quad J_\psi = -J_\phi^{bps} + \mathcal{O}(\beta_R^{-2}), \\ Q_1 &\simeq Q_1^{bps}, \quad Q_5 \simeq Q_5^{bps}, \quad Q_p = Q_p^{bps} + \mathcal{O}(\beta_R^{-2}).\end{aligned}\quad (3.18)$$

Note that the BPS charges satisfy (3.13). They are written in terms of the parameters that describe the system in (A.21). Finally, the expansion of the entropy yields

$$S = S^{bps} + \mathcal{O}(\beta_R^{-1}), \quad \text{with} \quad S^{bps} = 2\pi \left[Q_1^{bps} Q_5^{bps} Q_p^{bps} - (J_\phi^{bps})^2\right]^{1/2}. \quad (3.19)$$

With the above off-BPS expansion, we are ready to define the BPS chemical potentials. Comparing (3.16) with (3.3) we obtain,

$$\omega_{\phi,\psi} = \mp \frac{\pi J_\phi^{bps}}{\left[Q_1^{bps} Q_5^{bps} Q_p^{bps} - (J_\phi^{bps})^2\right]^{1/2}}, \quad \phi_i = \frac{\pi Q_1^{bps} Q_5^{bps} Q_p^{bps}}{Q_i^{bps} \left[Q_1^{bps} Q_5^{bps} Q_p^{bps} - (J_\phi^{bps})^2\right]^{1/2}}. \quad (3.20)$$

Notice that these chemical potentials only depend on the BPS conserved charges.

Now that all the BPS statistical mechanics conjugated pairs and entropy are defined, we are ready to obtain the other thermodynamic functions. For example, consider the quantum statistical relation,

$$I = \beta E - \beta \sum_{i=1,5,p} \Phi^{(i)} Q_i - \beta \sum_{j=\phi,\psi} \Omega^j J_j - S. \quad (3.21)$$

After the off-BPS expansion, *i.e.*, using (3.18), (3.19) and (3.16) it yields

$$I = \beta \left(E^{bps} - \sum_{i=1,5,p} Q_i^{bps} - \sum_{j=\phi,\psi} \Omega_{bps}^j J_j^{bps} \right) + \sum_{i=1,5,p} \phi_i Q_i^{bps} + \sum_{j=\phi,\psi} \omega_j J_j^{bps} - S_{bps} + \mathcal{O}(\beta_R^{-1}). \quad (3.22)$$

The term in between brackets vanishes due to the BPS relations (3.13) and (3.17). Then, taking $\beta \rightarrow \infty$, we are left with the supersymmetric quantum statistical relation (SQSR) for the three-charged BH,

$$I_{bps} = \phi_1 Q_1^{bps} + \phi_5 Q_5^{bps} + \phi_p Q_p^{bps} + 2\omega_\phi J_\phi^{bps} - S_{bps}, \quad (3.23)$$

where I_{bps} is the value of the Euclidean action in the supersymmetric limit of the D1-D5-P BH, and we used $J_\psi^{bps} = -J_\phi^{bps}$ and $\omega_\psi = -\omega_\phi$. Notice that I_{bps} corresponds to the Legendre transformation of the entropy with respect to all the BPS chemical potential and therefore should be interpreted as the BH free energy.

The off-BPS expansion of the horizon angular velocities and electric potentials gives the supergravity chemical potentials as the next to leading order term of the expansion around the BPS solution. The motivation for this expansion analysis comes from the fact that BPS BHs can be studied as dual configurations to supersymmetric ensembles at zero temperature *but* non-zero chemical potentials in the dual CFT [12]. The supergravity conjugated potentials (3.20) are then the strong coupling dual objects to the CFT chemical potentials. The SQSR relation (3.23) will be connected to the well-known Sen's entropy relation in the next section.

4 Euclidean zero-temperature and entropy functional formalisms

In previous sections we have described two apparently unrelated procedures to obtain the entropy of supersymmetric BH that naturally contain the definitions of pairs of conjugated variables, related to the BH charges. In this section we show that both procedures produce basically the same body of final definitions, even though conceptually both approaches are rather different.

That both approaches produce the same final chemical potentials and definitions can be seen in any of the examples at hand. As usual, the best way to illustrate our point is to pick a system that captures the fundamental ingredients, while avoiding features that do not play a key role and produce unnecessary distraction from the main point. In the present case, the appropriate system is the non-rotating D1-D5-P BH (later, we will discuss the rotating case). Comparing the thermodynamic relations (3.19), (3.20), and the Sen's relations (2.13), (2.14), we can indeed confirm that all the key quantities agree in the two formalisms. Explicitly we have that

$$\phi_i = 2\pi e_i, \quad Q_i = q_i, \quad I_{bps} = 2\pi f. \quad (4.1)$$

Nevertheless, that both frameworks are equivalent is *a priori* not at all obvious since they have important differences. Sen's approach relies completely on the structure of the near horizon geometry. In particular, the entropy is constructed analyzing Wald's prescription and Einstein equations in these spacetimes and all the analysis is carried on at the BPS bound *i.e.*, when the solution is extremal. In contrast, the zero temperature limit approach relies on the thermodynamical properties of BH and, in principle, uses the whole spacetime, not only the near horizon region. The resulting thermodynamic definitions come as a limiting behavior of non-extremal BH and have a nice straightforward interpretation in terms of the dual CFT thermodynamics.

4.1 Near-horizon and asymptotic contributions to the Euclidean action

To understand why the above close relations between the two formalisms hold, let us go back to the calculation of the Euclidean action for general BH in the *off*-BPS regime. Inspired in ten

dimensional type II supergravity, we start with the general action¹¹

$$I = \frac{1}{16\pi G} \int_{\Sigma} \sqrt{-g} \left(R - \frac{1}{2}(\partial\Psi)^2 - \frac{1}{2n!} e^{\alpha\Psi} F_{(n)}^2 \right) + \frac{1}{8\pi G} \int_{\partial\Sigma} K, \quad (4.2)$$

where Σ is the spacetime manifold, $\partial\Sigma$ the boundary of that manifold and K is the extrinsic curvature. In the BH case, once we have switched to Euclidean regime, it is necessary to compactify the time direction to avoid a conical singularity. This compactification defines the Hawking temperature as the inverse of the corresponding compactification radius.

To evaluate the Euclidean action on the BH solution, one of the methods to obtain a finite result, *i.e.*, to regularize and renormalize the action, consists of putting the BH in a box and subtract the action of a background vacuum solution (g^0, Ψ^0, F^0) . This procedure also defines the “zero” of all the conserved charges. For asymptotic flat solutions we use Minkowski, while for asymptotic AdS solutions we use AdS. Once in the box, the radial coordinate is restricted to the interval (r_+, r_b) , where r_+ is the position of the horizon and r_b corresponds to an arbitrary point which limits the box and that at the end is sent to infinity. Another important ingredient is the boundary conditions on the box. Basically, depending on which conditions we impose on the different fields, we will have fixed charges or fixed potentials. If we do not add any boundary term to the above action, we will be working with fixed potentials, *i.e.*, we will work in the grand canonical ensemble [28].

The field equations are derived from a variational principle, where fields are kept constant at the boundaries. In particular, the trace of the equation that comes from the variation of the metric (for the D1-D5-P system, see equation (A.2)) implies that

$$R - \frac{1}{2}(\partial\Psi)^2 = a e^{\alpha\Psi} F_{(n)}^2, \quad (4.3)$$

where a depends on the spacetime dimensions and n . Therefore, on-shell, the action reduces to¹²,

$$I = \frac{b}{8\pi G} \int_{\Sigma} e^{\alpha\Psi} F_{(n)}^2 + \frac{1}{8\pi G} \int_{\partial\Sigma} (K - K^0), \quad (4.4)$$

where b depends on the spacetime dimensions and n . The first term is a volume integral over Σ that can easily be converted into a boundary integral over $\partial\Sigma$, once we recall that we are considering electric fields only and hence $F_{(n)} = dC_{(n-1)}$. Integrating by parts we get

$$I = \frac{c}{8\pi G} \int_{\partial\Sigma} e^{\alpha\Psi} F_{(n)} C_{(n-1)} + \frac{1}{8\pi G} \int_{\partial\Sigma} (K - K^0), \quad (4.5)$$

where c depends on the spacetime dimensions and n . At this point, the on-shell Euclidean action is completely recasted in two surface integrals terms, evaluated at r_+ and r_b . Consider first the extrinsic curvature term. At r_b , we get βE_b , where E_b is the quasi-local energy. When r_b is taken to infinity, E_b reduces to the BH energy E and we recover usual term βE . At r_+ , only K contributes and gives minus the area of the horizon divided by $4G$, *i.e.*, minus the Bekenstein-Hawking entropy

¹¹For simplicity, the reasoning is done at the level of two derivative Lagrangian. Nevertheless, following Wald's approach for higher derivative actions, we notice that the BH action can always be recast as surface integrals. Moreover, for definiteness, we anchor our discussion to type II action, but whenever needed we make comments to extend our arguments to more general theories.

¹²Where we have used that the action of the background vacuum solution over Σ is zero.

S . Next consider the first term. Here the integral over time gives the factor β , while the integration over the other directions (of the induced metric determinant at the boundary times $e^{a\Psi} F_{(n)}$) gives the corresponding electric charge Q . Therefore, we get

$$\frac{c}{8\pi G} \int_{\partial\Sigma} e^{a\Psi} F_{(n)} C_{n-1} = -\beta Q [C_{(n-1)}(r_b) - C_{(n-1)}(r_+)] . \quad (4.6)$$

Then, we use the definition of the conjugated chemical potential ϕ as the difference of the gauge potential at infinity and at the horizon,

$$\Phi = C|_{\infty} - C|_{r_+} , \quad (4.7)$$

and hence, when r_b is sent to infinity, we recover the usual term $-\beta Q\Phi$. As a grand total we obtain the QSR,

$$I = \beta E - \beta Q\Phi - S . \quad (4.8)$$

Now, it is important to notice that the definition of Φ is gauge independent, and therefore we can always choose a particular gauge that simplifies the picture depending on which physical concepts we want to stress. Here, we choose the “natural gauge” adapted to the BPS limiting cases, $C|_{\infty} = \Phi_{bps}$, where Φ_{bps} is usually 1 in natural units and for asymptotically flat BHs. Note that in this gauge one has $C|_{r_+} = \Phi_{bps} - \Phi$. This gauge choice is the one that makes direct contact between the Euclidean zero temperature and entropy function formalisms for reasons that will become clear after (4.12).

At this point we are ready to rewrite the Euclidean action in two pieces, one evaluated in the first boundary at $r = r_+$, and the other in the second boundary at $r = \infty$,

$$I = \int_{r=r_+} \left\{ \frac{c}{8\pi G} e^{a\Psi} F_{(n)} C_{(n-1)} + \frac{1}{8\pi G} K \right\} + \int_{r=\infty} \left\{ \frac{c}{8\pi G} e^{a\Psi} F_{(n)} C_{(n-1)} + \frac{1}{8\pi G} (K - K^0) \right\} . \quad (4.9)$$

Evaluating both terms as we did before but now in the adapted gauge we get,

$$I = \underbrace{\beta(\Phi_{bps} - \Phi)Q - S}_{r=r_+} + \underbrace{\beta(E - \Phi_{bps}Q)}_{r=\infty} . \quad (4.10)$$

Therefore we can always find a gauge in which the Euclidean action splits in two contributions, one at the horizon and the other in the asymptotic region. It is perfectly adapted to understand the near horizon regime. Equally interesting, this expression is also adapted to understand the supersymmetric limit. In fact, from our discussion in section 3, it is easy to see that the first term exactly reproduces the SQSR, *i.e.*,

$$\lim_{BPS \text{ limit}} \beta(\Phi_{bps} - \Phi)Q - S = \phi Q_{bps} - S_{bps} . \quad (4.11)$$

On the other hand, the asymptotic term vanishes due to fact that $\Phi_{bps} = 1$, and thus the leading term in the expansion is nothing else than the BPS relation $E_{bps} = Q_{bps}$ characteristic of supersymmetric regimes, *i.e.*,¹³

$$\lim_{BPS \text{ limit}} \beta(E - \Phi_{bps}Q) = \lim_{BPS \text{ limit}} \beta(E - Q) = 0 . \quad (4.12)$$

¹³This discussion is strictly valid for the asymptotically flat BHs where $\Phi_{bps} = 1$. For asymptotically AdS BHs, the normalization usually chosen in the literature yields in general $\Phi_{bps} \neq 1$. However, in this case, the term inside brackets in (4.12) still vanishes because it is exactly the BPS constraint on the charges. This follows by construction and is explicitly confirmed for 5D gauged supergravity in [12, 15].

(Note that in the last equality we jump some steps that were already explained in detail after (3.6), and that we do not repeat here. They guarantee that this term indeed vanishes and does not give an indeterminacy of the type $\infty \cdot 0$). *We conclude that the Euclidean action of the BH at the BPS bound is given exclusively from the near horizon part of the solution.* This is another way to characterize the attractor mechanism, since the physical properties of the solution are captured entirely by the near horizon geometry. From the above result, it is easy to see why, for supersymmetric cases, I is related to f . First, both are functionals of the near horizon geometry alone. Also, the time and radial integrations are trivial and only integration on the other space directions actually contribute. In fact, this is a way to understand why in the definition of f there is no integration in the AdS part of the near horizon metric. Note also that in Sen's approach the f function is defined as the integral of the *string frame* Lagrangian evaluated at the near horizon geometry. Since in this geometry the dilaton is a constant, the string frame and Einstein frame Lagrangians are related by a trivial constant factor.

We now discuss the effects introduced by addition of rotation. Working in a coordinate system in which the geometry is not rotating at infinity, the action can be split as

$$I = \underbrace{\beta(\Phi_{bps} - \Phi)Q + \beta(\Omega_{bps} - \Omega)J - S}_{r = r_+} + \underbrace{\beta(E - \Phi_{bps}Q - \Omega_{bps}J)}_{r = \infty}. \quad (4.13)$$

By definition, the near horizon term contains all the information on the chemical potentials (once the BPS limit is taken),

$$\lim_{BPS \text{ limit}} \beta(\Phi_{bps} - \Phi)Q + \beta(\Omega_{bps} - \Omega)J - S = \phi Q_{bps} + \omega J_{bps} - S_{bps}, \quad (4.14)$$

while the asymptotic term is again the BPS constraint between the several charges and thus vanishes,

$$\lim_{BPS \text{ limit}} \beta(E - \Phi_{bps}Q - \Omega_{bps}J) = 0. \quad (4.15)$$

For asymptotically flat BHs one always has $\Omega_{bps} = 0$ and (4.15) reduces to (4.12). The horizon of flat BHs does not rotate (angular momentum comes from the Poynting vector of electromagnetic fields) and this is one way to understand why the angular momenta does not appear in their BPS constraint. On the other hand, the horizon velocity of asymptotically AdS BHs is, in general, non-vanishing, and thus the angular momenta also contributes to the BPS constraint of these BHs.

4.2 Relation between chemical potentials in the two formalisms

At this point only reminds to understand the relation between the conjugated potentials in both pictures. In Sen's approach, the information about them is contained in the electric fields of the near horizon geometry, while in the Euclidean zero temperature formalism this information is encoded in the next to leading order term in an off-BPS expansion of the full geometry. Although these definitions seem to be rather different at first sight, notice that in Sen's approach the field strength is just the radial derivative of the potential evaluated at the horizon. In the Euclidean zero temperature case, the off-BPS expansion can be rewritten as an expansion in the radial position of the horizon ρ_+ . Therefore, the next to leading order term in the off-BPS expansion of the gauge potential at ρ_+ is proportional to its derivative with respect to the radial position of the

horizon. Hence it is proportional to the field strength at the horizon. These words can be made very precise by taking an example. Consider the D1-D5-P BH we have been working with (again we do not include rotation in the analysis to avoid unnecessary non-insightful complications). In the full geometry (2.5), where the zero temperature limit procedure is applied, we work with the t, r coordinates. Sen's approach uses instead the near-horizon fields (2.6) or (2.8) described in terms of (τ, ρ) coordinates. The two set of coordinates are related by (2.7). Our purpose in the next lines is to understand the first relation in (4.1). For definiteness we focus on the relation $\phi_1 = 2\pi e_1$. From (2.5), $C_{ty} = -Ms_1 c_1/(\rho + Ms_1^2)$, and one also has the relation between the gauge field written in the two coordinate systems, $C_{ty} = \frac{\partial \tau}{\partial t} C_{\tau y}$. In the near-horizon approach, the expression for e_1 comes from the radial derivative of the potential evaluated at the BPS horizon ($\rho_+^{bps} = 0$):

$$e_1 = \left\{ \partial_\rho C_{\tau y} \right\}_{\rho=\rho_+^{bps}} = \frac{1}{2} \sqrt{\frac{Q_5^{bps} Q_p^{bps}}{Q_1^{bps}}}. \quad (4.16)$$

In the Euclidean zero temperature approach, the electric potential is obtained by contracting the gauge field with the timelike Killing vector $\xi = \partial_t$ yielding: $\Phi^{(1)} = -C_{ty}|_{\rho=\rho_+} = -\frac{\partial \tau}{\partial t} C_{\tau y}|_{\rho=\rho_+}$ (note that $\rho_+ = \rho_+^{bps} = 0$ only in the BPS case). As is clear from (3.16), our conjugated potential is defined as

$$\phi_1 = -\frac{1}{2} \left\{ \frac{\partial \Phi^{(1)}}{\partial \beta_R^{-1}} \right\}_{\beta_R^{-1}=0} = -\frac{1}{2} \frac{\partial \rho_+}{\partial \beta_R^{-1}} \left\{ \frac{\partial \Phi^{(1)}}{\partial \rho_+} \right\}_{\rho_+=\rho_+^{bps}}. \quad (4.17)$$

Note the following key relations,¹⁴

$$\left\{ \frac{\partial \Phi^{(1)}}{\partial \rho_+} \right\}_{\rho_+=\rho_+^{bps}} = -\frac{\partial \tau}{\partial t} \left\{ \partial_{\rho_+} [C_{\tau y}|_{\rho=\rho_+}] \right\}_{\rho_+=\rho_+^{bps}} = -\frac{1}{2} \frac{\partial \tau}{\partial t} \left\{ \partial_\rho C_{\tau y} \right\}_{\rho=\rho_+^{bps}}. \quad (4.18)$$

From (4.16)-(4.18), one finally has

$$\phi_1 = \frac{1}{4} \frac{\partial \rho_+}{\partial \beta_R^{-1}} \frac{\partial \tau}{\partial t} e_1 = 2\pi e_1. \quad (4.19)$$

The last equality follows from (2.7), and from $\rho_+ = M = 4\pi \sqrt{Q_1^{bps} Q_5^{bps} Q_p^{bps} \beta_R^{-1}}$ (see (3.15) and the last statement of Appendix A.2). Physically we can understand it by noting that the near-horizon coordinates are precisely the ones appropriate to find the value of the temperature, that avoids the standard conical singularity in the Euclidean near-horizon geometry. An analysis along the lines carried here for this specific case can be carried on for general cases and yield the relations (4.1) between the conjugated potentials found using the two formalisms.

To summarize, we have seen that for supersymmetric BH, the Euclidean action and all the chemical potentials are defined in the near horizon geometry. The asymptotic region would contribute only in off-BPS cases. We have also shown why the chemical potentials are proportional

¹⁴The presence of the factor 1/2 in the last equality is due to a subtlety that occurs when we take $\partial_{\rho_+} \Phi$ (and thus before sending $\rho_+ \rightarrow \rho_+^{bps}$). In the large δ_1 regime one has $Ms_1^2 \sim Q_1^{bps} - M/2$. Using this and $\rho_+ = M$ yields, in the denominator of Φ , $\rho_+ + Ms_1^2 \sim Q_1^{bps} + \rho_+/2$. This is the 1/2 that appears when we further take the ρ_+ derivative. Note that this factor does not appear in the last derivative of (4.18), $\partial_\rho C_{\tau y}$, because here we take the radial derivative evaluated on the on-shell solution $\rho = 0$.

to the electric fields in the near horizon region, and ultimately, we have understood, from the BH thermodynamics, the emergence of Sen's entropy function as the extremal limit of the quantum statistical relation or SQSR. As a bonus, we can now extend the statistical mechanics analysis like the SQSR to BPS solutions with no off-BPS known extension, because we have learned how to calculate the relevant chemical potentials directly in the BPS regime with no need of the limiting procedure.

5 Extremal (non-BPS) black holes

So far we have seen that two completely different procedures, namely the Euclidean zero temperature formalism and Sen's entropy formalism allow to compute the entropy and conjugated chemical potentials of *supersymmetric* BHs. This is not an accident as proved in the previous section. Now, as is well-known, Sen's approach also allows to find the attractor values of *non*-BPS extreme BHs [9, 26]. So a question that naturally raises is if whether or not the Euclidean zero temperature approach is also able to deal successfully with these type of solutions. In this section we address this issue.

It is straightforward to conclude that the Euclidean formalism indeed allows to find the chemical potentials of non-BPS extreme configurations. This follows from an analysis similar to the derivation presented in section 4, but this time slightly modified to account for the fact that the extreme BH is not BPS. Choosing the gauge $C|_{\infty} = \Phi_{ext}$ (and thus $C|_{r_+} = \Phi_{ext} - \Phi$), the extreme analogue of (4.14) is

$$I = \underbrace{\beta(\Phi_{ext} - \Phi)Q + \beta(\Omega_{ext} - \Omega)J - S}_{r = r_+} + \underbrace{\beta(E - \Phi_{ext}Q - \Omega_{ext}J)}_{r = \infty}. \quad (5.1)$$

where the first term boils down to the extreme counterpart of (4.14),

$$\lim_{ext. \text{ limit}} \beta(\Phi_{ext} - \Phi)Q + \beta(\Omega_{ext} - \Omega)J - S = \phi Q_{ext} + \omega J_{ext} - S_{ext}, \quad (5.2)$$

containing all the information on the chemical potentials.

On the other hand, for non-BPS extreme solutions, we find that the asymptotic term in (5.1),

$$\lim_{ext. \text{ limit}} \beta(E - \Phi_{ext}Q - \Omega_{ext}J), \quad (5.3)$$

in general does not vanish, as oppose to its BPS cousin. However, we find the following important feature, at least in the cases we studied: i) the cases where (5.3) does not vanish correspond to extreme rotating solutions that have in common the presence of an ergoregion; (ii) rotating extreme solutions without ergosphere and non-rotating extreme solutions have vanishing (5.3). This occurs at least on the three-charged, four-charged and Kerr-Newman systems. In the cases where it vanishes we again have that the Euclidean action of the BH at the extreme bound is given exclusively from the near horizon part of the solution. The physical properties of the solution are captured entirely by the near horizon geometry, which makes the attractor mechanism manifest ¹⁵.

¹⁵This discussion is at the level of two derivative Lagrangian. If corrections are added, we expect that the asymptotic part vanishes producing a finite result, also for extreme BH with ergoregion.

In the above extremal non-BPS cases, we can explicitly verify that the two formalisms indeed yield the same results. For this exercise and as an example, we will discuss below two extreme three-charged BH (whose BPS cousin was studied in the previous sections). To emphasize that the relation between the Euclidean and Sen's formalism is universal and not restricted to the three-charged system, in Appendices B.1 and B.2, we further extend the exercise to three other non-trivial extreme solutions whose properties have been studied within Sen's formalism.

5.1 Extreme three-charged black hole with ergoregion

In the D1-D5-P solution described by (A.4)-(A.8) we can take, instead of the BPS limit described in section 3, a different limit that yields an extreme (but not BPS) BH with an ergoregion. This is a case in which the system shows only partial attractor mechanism.

Concretely, we take the near-extreme limit

$$M \rightarrow (a_1 + a_2)^2 + \varepsilon, \quad \varepsilon \ll 1. \quad (5.4)$$

When the off-extreme parameter ε vanishes, the temperature indeed vanishes since $\beta_R \rightarrow \infty$ in (3.9). The off-extreme expansion of the conserved charges (3.8) around the corresponding extreme values (obtained by replacing M by $(a_1 + a_2)^2$ in (3.8)) is straightforward, and the expansion of the thermodynamic quantities (3.9)-(3.12) yields

$$\begin{aligned} \beta_L &= \pi (c_1 c_5 c_p - s_1 s_5 s_p) \frac{(a_1 + a_2)^2}{\sqrt{a_1 a_2}} + \mathcal{O}(\varepsilon), & \beta_R &= 2\pi (a_1 + a_2)^2 (c_1 c_5 c_p + s_1 s_5 s_p) \frac{1}{\sqrt{\varepsilon}}, \\ S &= S^{ext} + \mathcal{O}(\beta_R^{-1}), & \Omega^{\phi, \psi} &= \Omega_{ext}^{\phi, \psi} - \frac{2\omega_{\phi, \psi}}{\beta_R} + \mathcal{O}(\beta_R^{-2}), \\ \Phi^{(i)} &= \Phi_{ext}^{(i)} - \frac{2\phi_i}{\beta_R} + \mathcal{O}(\beta_R^{-2}), & i &= 1, 5, p, \end{aligned} \quad (5.5)$$

where the extreme values satisfy

$$\begin{aligned} S_{ext} &= 2\pi \sqrt{a_1 a_2} (a_1 + a_2)^2 (c_1 c_5 c_p + s_1 s_5 s_p), \\ \Omega_{ext}^{\phi} &= \Omega_{ext}^{\psi} = -[(a_1 + a_2)(c_1 c_5 c_p + s_1 s_5 s_p)]^{-1}, \\ \Phi_{ext}^{(i)} &= \frac{(\tanh \delta_i) c_1 c_5 c_p + (\coth \delta_i) s_1 s_5 s_p}{c_1 c_5 c_p + s_1 s_5 s_p}, \quad i = 1, 5, p, \end{aligned} \quad (5.6)$$

and the conjugated potentials are

$$\begin{aligned} \omega_{\phi, \psi} &= -\frac{\pi}{2} \left[\frac{a_1 + a_2}{\sqrt{a_1 a_2}} \frac{c_1 c_5 c_p - s_1 s_5 s_p}{c_1 c_5 c_p + s_1 s_5 s_p} \pm (a_1 - a_2) \sqrt{a_1 a_2} \right], \\ \phi_i &= -\frac{\beta_L}{s_i c_i} [\tanh \delta_1 \tanh \delta_5 \tanh \delta_p - \coth \delta_1 \coth \delta_5 \coth \delta_p]^{-1}, \quad i = 1, 5, p. \end{aligned} \quad (5.7)$$

These expressions for the potentials could be rewritten only in terms of the conserved charges as expected by the attractor mechanism. We avoid doing it because the expressions are long and non-insightful.

The QSR for this system is

$$\begin{aligned}
I = & \beta \left(E^{ext} - \sum_{i=1,5,p} \Phi_{ext}^{(i)} Q_i^{ext} - \Omega_{ext}^\phi J_{ext}^\phi - \Omega_{ext}^\psi J_{ext}^\psi \right) \\
& + \sum_{i=1,5,p} \phi_i Q_i^{ext} + \omega_\phi J_{ext}^\phi + \omega_\psi J_{ext}^\psi - S_{ext} + \mathcal{O}(\beta_R^{-1}) .
\end{aligned} \tag{5.8}$$

In the supersymmetric system the analogue of the term in between brackets vanishes due to the BPS constraint on the conserved charges. But, in general, for non-BPS extreme BHs it does not vanish (see discussion associated with (5.3)). In the present case the factor in between brackets is $\frac{(a_1+a_2)^2}{2} \frac{c_1 c_5 c_p - s_1 s_5 s_p}{c_1 c_5 c_p + s_1 s_5 s_p}$. Note that this quantity vanishes when rotation is absent ($a_1 = a_2 = 0$). When it is present, the solution has an ergoregion and the non-vanishing contribution is associated with its existence. Notice that in this case the Euclidean action is not well-defined but, nevertheless, the chemical potentials (5.8) take finite values and are physically relevant.

5.2 Extreme three-charged black hole without ergoregion

The metric of the D1-D5-P system is also a solution of type I supergravity. A fundamental difference between type IIB and type I theories is that the later theory has half of the supersymmetries of type IIB. This feature implies that in type I, if we reverse the sign of the momentum in the BPS D1-D5-P black hole, we get a distinct solution that is extreme but non-BPS. We study this solution of type I in this subsection, as the main example of an extreme non-BPS solution without ergoregion where attractor mechanism is fully manifest.

The near-extreme limit we now consider is similar to the near-BPS limit (3.14) in which we send the boosts to infinity; the difference being that now we take one of the boosts to be *negative* (again, by U -dualities it does not matter which one). The reason why these two limits are indeed different and, in particular, why one of them yields a BPS BH and the other not is the following [25]. The three-charged BH describes, in the supergravity approximation and after dualities, the F1-NS5-P system that is a configuration of heterotic string theory compactified on $T^4 \times S^1$. We can describe this system as an effective fundamental string with winding number $n_1 n_5$ (where n_1, n_5 are the numbers of $F1$ and $NS5$ constituents), and with momentum excitations traveling along it. Now, heterotic string theory is chiral. Hence, the direction of the momentum along the fundamental string sets if the solution is supersymmetric or not. In our conventions, the supersymmetric configuration F1-NS5-P is the one with no right-movers. So, in the supergravity approximation, the BPS BH that describes this system is obtained by taking $\delta_p \rightarrow +\infty$. But we can also consider the heterotic string configuration with only right-movers. Due to the chirality property, this F1-NS5-P configuration is then not supersymmetric. And the corresponding supergravity solution obtained by taking $\delta_p \rightarrow -\infty$ is not a BPS BH. Note that this solution is however extreme, *i.e.*, it has zero temperature. The reason being that there are no left-movers to collide with the right-movers and generate the closed string emission that describes the Hawking radiation.

So we take the near-extreme limit ($\delta_{1,5} > 0$; $\delta_p < 0$, $Q_p < 0$)¹⁶:

$$J_\phi - J_\psi \rightarrow 0; \quad M \rightarrow 0, \quad \delta_{1,5} \rightarrow \infty, \quad Q_{1,5} \text{ fixed}; \quad \delta_p < 0 \text{ finite}. \quad (5.10)$$

The conserved charges of the non-extreme three-charged BH are listed in (3.8), and the temperature, entropy, and angular velocities and potentials at the horizon are given in (3.9)-(3.11).

The charges in the extreme solution satisfy the constraint

$$E^{ext} = Q_1^{ext} + Q_5^{ext} - Q_p^{ext}, \quad J_\psi^{ext} = J_\phi^{ext}, \quad (5.11)$$

where we used $Q_p^{ext} = -Me^{-2\delta_p}/4$.

The off-extreme parameter, $\varepsilon = Me^{2\delta_p}/4$, measures energy above extremality and is such that $E \equiv E^{bps} + \varepsilon$. The expansion of the left and right temperatures in terms of the off-extreme parameter ε yields,

$$\beta_L = \pi \sqrt{Q_1^{ext} Q_5^{ext}} \frac{1}{\sqrt{\varepsilon}}, \quad \beta_R = \pi Q_1^{ext} Q_5^{ext} [-Q_1^{ext} Q_5^{ext} Q_p^{ext} - (J_\phi^{ext})^2]^{-1/2}. \quad (5.12)$$

The extreme limit corresponds to send the temperature $T \rightarrow 0$ by sending $\beta_L \rightarrow \infty$ while keeping β_R finite. In this limit there are no left-movers, only right-movers. The first relation in (5.12) defines ε in terms of β_L .

The expansion for the relevant thermodynamic quantities is

$$\begin{aligned} S &= S_{ext} + \mathcal{O}(\beta_L^{-1}), \quad \Omega^{\phi,\psi} = \Omega_{ext}^{\phi,\psi} - \frac{2\omega_{\phi,\psi}}{\beta_L} + \mathcal{O}(\beta_L^{-2}), \\ \Phi^{(i)} &= \Phi_{ext}^{(i)} - \frac{2\phi_i}{\beta_L} + \mathcal{O}(\beta_L^{-2}), \quad i = 1, 5, p, \end{aligned} \quad (5.13)$$

where

$$\begin{aligned} S_{ext} &= 2\pi [-Q_1^{ext} Q_5^{ext} Q_p^{ext} - (J_\phi^{ext})^2]^{1/2}, \\ \Omega_{ext}^{\phi,\psi} &= 0, \quad \Phi_{ext}^{(1,5)} = 1, \quad \Phi_{ext}^{(p)} = -1. \end{aligned} \quad (5.14)$$

The conjugated potentials are

$$\begin{aligned} \omega_{\phi,\psi} &= -\frac{\pi J_\phi^{ext}}{[-Q_1^{ext} Q_5^{ext} Q_p^{ext} - (J_\phi^{ext})^2]^{1/2}}, \\ \phi_i &= -\frac{\pi Q_1^{ext} Q_5^{ext} Q_p^{ext}}{Q_i^{bps} [-Q_1^{ext} Q_5^{ext} Q_p^{ext} - (J_\phi^{ext})^2]^{1/2}}, \quad i = 1, 5, p. \end{aligned} \quad (5.15)$$

¹⁶ The rotation parameters in this limit go as

$$a_{1,2} = -\sqrt{M} \frac{J_\phi^{ext}}{2\sqrt{-Q_1^{ext} Q_5^{ext} Q_p^{ext}}} [1 + \mathcal{O}(\varepsilon)]. \quad (5.9)$$

For comparison, in the BPS limit $a_{1,2}$ go instead as (A.24).

Although this is a non-BPS solution, it satisfies the extremal constraint (5.11) that is linear in the charges. This, together with (5.14), has the consequence that (5.3) applied to this system vanishes, and the QSR for this system simplifies to

$$I_{ext} = \sum_{i=1,5,p} \phi_i Q_i^{ext} + 2\omega_\phi J_\phi^{ext} - S_{ext}, \quad (5.16)$$

where we used $J_\psi^{ext} = J_\phi^{ext}$ and $\omega_\psi = \omega_\phi$.

So, contrarily to the example of the previous subsection, where the system showed only partial attractor mechanism due to the existence of an ergoregion, in the present system the ergoregion is absent and the attractor mechanism is fully manifest, even though the extreme solution is not BPS.

6 Discussion

First of all, we would like to stress again the logic behind our approach: zero temperature limits to reach extremal configurations are naturally defined in statistical analysis of quantum field theories. The AdS/CFT correspondence then requires that there has to be a dual analysis for strings in AdS. Supergravity is just the tree level part of the above theory, and thus we do not expect in general a well defined zero temperature limit at this level. Here, by well defined we mean a limit that generates a finite Euclidean action when $T \rightarrow 0$. Nevertheless, we have found extremal BH that seem to be protected, and therefore have a well defined zero temperature limit. In some of these cases, the protection is based on supersymmetric arguments but, in other cases, we just have extremal non-BPS BH where in fact it is not well understood why supergravity is giving the correct answer.

In this article we have applied the Euclidean zero temperature formalism to supergravity solutions where Sen's formalism is well understood. In doing so, we have shown that this method agrees with Sen's entropy formalism, producing the same statistical mechanics functions like the entropy and the chemical potentials. On the top of this, the Euclidean zero temperature formalism has the key advantage of connecting the entropy functional with the statistical mechanics of the dual CFT and with the more canonical BH thermodynamics.

More concretely, due to the attractor mechanism, we found that the Euclidean action is itself given by the near horizon geometry alone, and therefore can be connected to Sen's approach to calculate the entropy. We showed how to relate all the different definitions in both approaches and why they match. In particular, we are able to understand the CFT dual of Sen's approach, using the established map for the corresponding quantum statistical relation. For example, Sen's function f (evaluated on-shell) is nothing more than the BPS limit of the Euclidean action and therefore is related to the dual CFT partition function. The above relation is relevant for the OSV conjecture [3], since now I_{bps} or f naturally takes the place of free energy of the supersymmetric BH.

We also worked out the extension to extremal but non-supersymmetric BH. Here, since we are dealing with two derivative Lagrangians, we divide BH in two groups: those with ergoregion and those without it. In all the cases with no ergoregion we have checked, the zero temperature limit produces a well defined QSR at extremality, where all the chemical potentials, entropy and the Euclidean action are related to Sen's approach. This is not a triviality, since here there is

no supersymmetry to protect these tree-level results. This result seems to imply a protection mechanism in the extremal case, as suggested in [25].

In the other case of extremal BH with ergoregions, we found an ill-defined limit, where the asymptotic contribution to the Euclidean action diverges. Nevertheless, the near horizon contribution is well behaved and produces the correct entropy and chemical potentials. These results are in agreement with Sen's approach since these geometries are not fully attracted. Therefore they depend also on asymptotic values of the moduli. We interpret this result as a confirmation that these geometries do receive corrections from string theory that in turn will modify the asymptotic region, and thus asymptotic charges like the energy. In fact, in [29] rotating BH of this sort were studied finding that for the ergoregion branch, the entropy, but not the energy, could be matched with the microscopic CFT.

We would like to point out that although we worked with standard low-energy supergravity, the inclusion of higher derivative terms should not spoil the results. In the Euclidean approach, one now has to compute the Euclidean action with the modified Lagrangian and define the entropy as its Legendre transform with respect to the BPS chemical potentials. This should give the same entropy as defined by Wald (see [30]).

The zero temperature limit analysis of supersymmetric CFT ensembles motivated the corresponding analysis in the dual supergravity system. In this paper our main goal was to make direct contact between this formalism and Sen's entropy function approach. The Euclidean zero temperature formalism further allows to scan the phase structure of BH. A paradigm on the useful information that this formalism allows to find about the CFT living on the boundary of a BH geometry can be found in [12, 13]. It would be interesting to make a similar application, this time to study the CFT of the BH systems discussed in this paper.

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Appendices

A Three-charged black hole: solution and thermodynamics

A.1 The D1-D5-P black hole

In this section we describe the D1-D5-P BH and its thermodynamic properties that are used in sections 3-5. The most general solution with arbitrary charges was originally constructed in [22] (see also [23]). This solution generalizes the case with equal D1 and D5 charges found previously in [32] and whose BPS limit yields the BMPV BH [33]. Here we follow the notation of [23, 31].

This three-charged BH is a solution of type IIB supergravity. The only IIB fields that are turned on are the graviton $g_{\mu\nu}$, the dilaton Ψ , and the RR 2-form $C \equiv C_{(2)}$. For the field strength one has simply $F_{(3)} = dC_{(2)}$ since the RR field $C_{(0)}$ and the NSNS field $H_{(3)}$ are absent. The type IIB action, in the Einstein frame, reduces in these conditions to

$$I = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left[R - \frac{1}{2} \partial_\mu \Psi \partial^\mu \Psi - \frac{1}{12} e^\Psi F_{(3)}^2 \right], \quad (\text{A.1})$$

where g is the determinant of the Einstein metric. The field equations that follow from variation of action (A.1) are

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \left(\partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu} \partial_\sigma \Psi \partial^\sigma \Psi \right) - \frac{1}{12} e^\Psi \left(3 F_{\mu\alpha\beta} F_\nu{}^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} F_3^2 \right) &= 0, \\ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Psi) - \frac{1}{12} e^\Psi F_3^2 &= 0, \\ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e^\Psi F^{\mu\alpha\beta}) &= 0. \end{aligned} \quad (\text{A.2})$$

Contraction of the graviton field equation yields for the Ricci scalar,

$$R = \frac{1}{2} \partial_\mu \Psi \partial^\mu \Psi + \frac{1}{24} e^\Psi F_3^2. \quad (\text{A.3})$$

The graviton in the Einstein frame is (the relation between the parameters describing the solution and the conserved charges is displayed in (3.8))

$$\begin{aligned} ds^2 &= -\frac{f}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} (dt^2 - dy^2) + \frac{M}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} (s_p dy - c_p dt)^2 \\ &+ \tilde{H}_1^{1/4} \tilde{H}_5^{3/4} \left(\frac{r^2 dr^2}{(r^2 + a_1^2)(r^2 + a_2^2) - Mr^2} + d\theta^2 \right) \\ &+ \frac{\tilde{H}_1 \tilde{H}_5 - (a_2^2 - a_1^2)(\tilde{H}_1 + \tilde{H}_5 - f) \cos^2 \theta}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} \cos^2 \theta d\psi^2 \\ &+ \frac{\tilde{H}_1 \tilde{H}_5 + (a_2^2 - a_1^2)(\tilde{H}_1 + \tilde{H}_5 - f) \sin^2 \theta}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} \sin^2 \theta d\phi^2 + \frac{M}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} (a_1 \cos^2 \theta d\psi + a_2 \sin^2 \theta d\phi)^2 \\ &+ \frac{2M \cos^2 \theta}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} \left[(a_1 c_1 c_5 c_p - a_2 s_1 s_5 s_p) dt + (a_2 s_1 s_5 c_p - a_1 c_1 c_5 s_p) dy \right] d\psi \\ &+ \frac{2M \sin^2 \theta}{\tilde{H}_1^{3/4} \tilde{H}_5^{1/4}} \left[(a_2 c_1 c_5 c_p - a_1 s_1 s_5 s_p) dt + (a_1 s_1 s_5 c_p - a_2 c_1 c_5 s_p) dy \right] d\phi + \tilde{H}_1^{1/4} \tilde{H}_5^{-1/4} \sum_{j=1}^4 dz_j^2, \end{aligned} \quad (\text{A.4})$$

where y is the coordinate on S^1 , and z_j 's ($j = 1, \dots, 4$) are the coordinates on the torus T^4 . We use the notation $c_i \equiv \cosh \delta_i$, $s_i \equiv \sinh \delta_i$, and

$$\begin{aligned} f(r) &= r^2 + a_1^2 \sin^2 \theta + a_2^2 \cos^2 \theta, & \tilde{H}_i(r) &= f(r) + M s_i^2, \text{ with } i = 1, 5, \\ g(r) &= (r^2 + a_1^2)(r^2 + a_2^2) - Mr^2. \end{aligned} \quad (\text{A.5})$$

The roots of $g(r)$, r_+ and r_- , are given by

$$r_{\pm}^2 = \frac{1}{2} (M - a_1^2 - a_2^2) \pm \frac{1}{2} \sqrt{(M - a_1^2 - a_2^2)^2 - 4a_1^2 a_2^2}, \quad (\text{A.6})$$

The system describes a regular BH¹⁷ when $r_+^2 > 0$, *i.e.*, for $M \geq (a_1 + a_2)^2$. The ten-dimensional determinant in the Einstein frame is $\sqrt{-g} = r \sin \theta \cos \theta \tilde{H}_1^{1/4} \tilde{H}_5^{3/4}$. The dilaton Ψ and 2-form RR gauge potential C which support the D1-D5-P configuration are

$$e^{2\Psi} = \frac{\tilde{H}_1}{\tilde{H}_5}, \quad (\text{A.7})$$

$$C_{(2)} = \frac{M}{\tilde{H}_1} \left[\cos^2 \theta (a_{t\psi} dt + a_{y\psi} dy) \wedge d\psi + \sin^2 \theta (a_{t\phi} dt + a_{y\phi} dy) \wedge d\phi \right. \\ \left. - s_1 c_1 dt \wedge dy - s_5 c_5 (r^2 + a_2^2 + M s_1^2) \cos^2 \theta d\psi \wedge d\phi \right], \quad (\text{A.8})$$

where we defined

$$\begin{aligned} a_{t\phi} &= a_1 c_1 s_5 c_p - a_2 s_1 c_5 s_p, & a_{t\psi} &= a_2 c_1 s_5 c_p - a_1 s_1 c_5 s_p, \\ a_{y\phi} &= a_2 s_1 c_5 c_p - a_1 c_1 s_5 s_p, & a_{y\psi} &= a_1 s_1 c_5 c_p - a_2 c_1 s_5 s_p. \end{aligned} \quad (\text{A.9})$$

By electric-magnetic duality¹⁸,

$$e^\Psi \sqrt{-g} F_{(3)}^{\mu_1 \mu_2 \mu_3} = \frac{1}{7!} \epsilon^{\mu_1 \mu_2 \mu_3 \nu_1 \dots \nu_7} F_{\nu_1 \dots \nu_7}^{(7)}, \quad (\text{A.10})$$

our configuration can be equivalently described either by the 2-form $C_{(2)}$ in (A.8) or by the 6-form $C_{(6)}$ that follows from (A.10). Using this equivalence, we rewrite (A.8) as

$$\begin{aligned} C_{(2)} &= -\frac{M}{\tilde{H}_1} (s_1 c_1 dt + a_{y\phi} \sin^2 \theta d\phi + a_{y\psi} \cos^2 \theta d\psi) \wedge dy, \\ C_{(6)} &= -\frac{M}{\tilde{H}_5} (s_5 c_5 dt + a_{t\psi} \sin^2 \theta d\phi + a_{t\phi} \cos^2 \theta d\psi) \wedge dy \wedge dz_1 \wedge dz_2 \wedge dz_3 \wedge dz_4. \end{aligned} \quad (\text{A.11})$$

The advantage of (A.11) is that we clearly identify the $C_{(2)}$ gauge potential sourced by the D1-brane charges and the $C_{(6)}$ field sourced by the D5-brane charges. Thus, this expression will be appropriate to find the electric potentials associated with the two type of D-branes. Note that all the $C_{\mu\nu}^{(2)}$ components contain the y -coordinate that parametrizes the S^1 , while all the $C_{\mu\nu\alpha\beta\gamma\sigma}^{(6)}$ components contain the y -coordinate and the z_j 's coordinates that parametrize the torus T^4 . This reflects the fact that D1-branes wrap S^1 and the D5-branes wrap the full internal space $T^4 \times S^1$.

¹⁷For $r_+^2 < 0$, *i.e.*, $M \leq (a_1 - a_2)^2$ the system can describe a smooth soliton without horizon [31, 34]. We will not discuss this solution.

¹⁸We use the convention $\epsilon^{tr\theta\phi\psi y z_1 z_2 z_3 z_4} = 1$, and the relation (A.10) is valid in the Einstein frame.

The general procedure to compute angular velocities when the geometry has several momenta can be found in [35]. Applied to our case, the angular velocities at the horizon along the ϕ -plane, Ω^ϕ , the ψ -plane, Ω^ψ , and the velocity along y , $\Phi^{(p)}$ are¹⁹

$$\begin{aligned}\Omega^\phi &= \frac{g_{ty}(g_{y\phi}g_{\psi\psi} - g_{y\psi}g_{\phi\psi}) + g_{t\phi}(g_{y\psi}^2 - g_{yy}g_{\psi\psi}) + g_{t\psi}(g_{yy}g_{\phi\psi} - g_{y\phi}g_{yy})}{g_{yy}g_{\phi\phi}g_{\psi\psi} + 2g_{y\phi}g_{y\psi}g_{\phi\psi} - g_{y\psi}^2g_{\phi\phi} - g_{y\phi}^2g_{\psi\psi} - g_{\phi\psi}^2g_{yy}} \Big|_{r=r_+}, \\ \Omega^\psi &= \frac{g_{ty}(g_{y\psi}g_{\phi\phi} - g_{y\phi}g_{\phi\psi}) + g_{t\phi}(g_{yy}g_{\phi\psi} - g_{y\phi}g_{yy}) + g_{t\psi}(g_{y\phi}^2 - g_{yy}g_{\phi\phi})}{g_{yy}g_{\phi\phi}g_{\psi\psi} + 2g_{y\phi}g_{y\psi}g_{\phi\psi} - g_{y\psi}^2g_{\phi\phi} - g_{y\phi}^2g_{\psi\psi} - g_{\phi\psi}^2g_{yy}} \Big|_{r=r_+}, \\ \Phi^{(p)} &= \frac{g_{ty}(g_{\phi\psi}^2 - g_{\phi\phi}g_{\psi\psi}) + g_{t\phi}(g_{y\phi}g_{\psi\psi} - g_{y\psi}g_{\phi\psi}) + g_{t\psi}(g_{y\psi}g_{\phi\phi} - g_{y\phi}g_{\phi\psi})}{g_{yy}g_{\phi\phi}g_{\psi\psi} + 2g_{y\phi}g_{y\psi}g_{\phi\psi} - g_{y\psi}^2g_{\phi\phi} - g_{y\phi}^2g_{\psi\psi} - g_{\phi\psi}^2g_{yy}} \Big|_{r=r_+},\end{aligned}\tag{A.12}$$

which yields

$$\begin{aligned}\Omega^\phi &= -\frac{a_2 r_+^2}{(r_+^2 + a_2^2)(r_+^2 c_1 c_5 c_p + a_1 a_2 s_1 s_5 s_p)}, \\ \Omega^\psi &= -\frac{a_1 r_+^2}{(r_+^2 + a_1^2)(r_+^2 c_1 c_5 c_p + a_1 a_2 s_1 s_5 s_p)}, \\ \Phi^{(p)} &= \frac{r_+^2 c_1 c_5 s_p + a_1 a_2 s_1 s_5 c_p}{r_+^2 c_1 c_5 c_p + a_1 a_2 s_1 s_5 s_p}.\end{aligned}\tag{A.13}$$

The horizon angular velocities are constant and, in particular have no angular dependence, as required by Carter's rigidity property of Killing horizons²⁰. The electric potentials at the horizon associated with the Q_1 and Q_5 gauge charges are computed using

$$\begin{aligned}\Phi^{(i)} &= -C_{\mu\{x^{(i)}\}}\xi^\mu \Big|_{r=r_+} \\ &= -\left[C_{t\{x^{(i)}\}} + C_{\phi\{x^{(i)}\}}\Omega^\phi + C_{\psi\{x^{(i)}\}}\Omega^\psi\right]_{r=r_+}, \quad i = 1, 5,\end{aligned}\tag{A.14}$$

where,

$$\xi = \partial_t + \Omega^\phi \partial_\phi + \Omega^\psi \partial_\psi,\tag{A.15}$$

is the null Killing vector generator of the horizon ($\Omega^{\phi,\psi}$ are the horizon angular velocities). We use the notation $\{x^{(1)}\} \equiv y$, the coordinate of S^1 wrapped by D1-branes, and $\{x^{(5)}\} \equiv yz_1z_2z_3z_4$,

¹⁹We identify $\Omega^y \equiv \Phi^{(p)}$ because the KK momentum plays effectively the role of a gauge charge with associated electric potential.

²⁰Note that the angular velocities can be more easily computed using the standard formulas valid for solutions rotating along a single axis, as long as we evaluate them at a specific θ coordinate. More concretely, an inspection of (A.12) concludes that the following relations are valid and provide the quickest computation of the corresponding quantities:

$$\Omega^\phi \equiv \frac{g_{t\phi}}{g_{\phi\phi}} \Big|_{r=r_+, \theta=0}, \quad \Omega^\psi \equiv \frac{g_{t\psi}}{g_{\psi\psi}} \Big|_{r=r_+, \theta=\pi/2}, \quad \Phi^{(p)} \equiv \frac{g_{ty}}{g_{yy}} \Big|_{r=r_+, \theta=0}.$$

the coordinates of $S^1 \times T^4$ wrapped by D5-branes. Using the C gauge potential written in (A.11) yields

$$\Phi^{(i)} = \frac{r_+^2 (\tanh \delta_i) c_1 c_5 c_p + a_1 a_2 (\coth \delta_i) s_1 s_5 s_p}{r_+^2 c_1 c_5 c_p + a_1 a_2 s_1 s_5 s_p}, \quad i = 1, 5. \quad (\text{A.16})$$

The temperature of the BH is $T = \kappa_h / (2\pi)$ where the surface gravity of the horizon is $\kappa_h^2 = -\frac{1}{2}(\nabla^\mu \xi^\nu)(\nabla_\mu \xi_\nu)|_{r=r_+}$, and ξ^μ is the Killing vector horizon generator defined in (A.15). The inverse temperature $\beta = 1/T$ is then

$$\beta = \frac{2\pi (r_+^2 + a_1^2) (r_+^2 + a_2^2)}{r_+ (r_+^4 - a_1^2 a_2^2)} (r_+^2 c_1 c_5 c_p + a_1 a_2 s_1 s_5 s_p). \quad (\text{A.17})$$

The entropy S is just horizon area (in the Einstein frame) divided by $4G_{10}$,

$$S = \frac{2\pi (r_+^2 + a_1^2) (r_+^2 + a_2^2)}{r_+^3} (r_+^2 c_1 c_5 c_p + a_1 a_2 s_1 s_5 s_p). \quad (\text{A.18})$$

To conclude this section, note that action (A.1) can be written in the string frame through the Weyl rescaling of the metric, $\tilde{g}_{AB} = e^{\Psi/2} g_{AB}$, yielding

$$I = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-\tilde{g}} \left[e^{-2\Psi} \left(\tilde{R} - 4\partial_\mu \Psi \partial^\mu \Psi \right) - \frac{1}{2 \cdot 3!} F_{(3)}^2 \right]. \quad (\text{A.19})$$

A.2 The near-BPS limit of the D1-D5-P black hole

In this appendix we present the detailed computation of the near-BPS limit of the D1-D5-P BH, and of the off-BPS construction that takes (3.9)-(3.11) into (3.15)-(3.23).

Using the trigonometric properties

$$c_i = \frac{e^{\delta_i} + e^{-\delta_i}}{2}, \quad s_i = \frac{e^{\delta_i} - e^{-\delta_i}}{2}, \quad (\text{A.20})$$

the gauge charges and ADM mass (3.8) are, in the near-BPS regime (3.14),

$$\begin{aligned} Q_p &= \frac{M e^{2\delta_p}}{4} - \frac{M e^{-2\delta_p}}{4} \equiv Q_p^{bps} - \varepsilon \\ Q_1 &\simeq \frac{M e^{2\delta_1}}{4} \equiv Q_1^{bps}, \quad Q_5 \simeq \frac{M e^{2\delta_5}}{4} \equiv Q_5^{bps}, \\ E &\simeq \left(Q_1^{bps} + Q_5^{bps} + Q_p^{bps} \right) + \varepsilon = E^{bps} + \varepsilon, \end{aligned} \quad (\text{A.21})$$

where the BPS constraint (3.13) was used. We can interpret the quantity $\frac{M e^{2\delta_p}}{4}$ as the number of left-movers, and $\varepsilon = \frac{M e^{-2\delta_p}}{4}$ as the number of right-movers (in the KK momentum sector). The BPS configuration, $\varepsilon = 0$, is the one with no right-movers. In the D1 and D5 sectors there are only left-movers since $\delta_{1,5} \rightarrow \infty$. From the last relation in (A.21), we conclude that ε is also an off-BPS parameter that measures energy above extremality. We can also rewrite $\varepsilon = \frac{M e^{-2\delta_p}}{4}$ as $M = 4\sqrt{Q_p^{bps}} \sqrt{\varepsilon}$, an expression that will be useful below.

The near-BPS limit (3.14) is completed with the angular momenta condition. It can be understood as follows. Inversion of (3.8) yields

$$a_1 = -\frac{1}{M} \frac{J_\psi c_1 c_5 c_p + J_\phi s_1 s_5 s_p}{c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2}, \quad a_2 = -\frac{1}{M} \frac{J_\phi c_1 c_5 c_p + J_\psi s_1 s_5 s_p}{c_1^2 c_5^2 c_p^2 - s_1^2 s_5^2 s_p^2}. \quad (\text{A.22})$$

In the near-BPS limit (3.14) one has $c_{1,5}^2 \simeq \frac{Q_{1,5}^{\text{bps}}}{M} + \frac{1}{2}$, $s_{1,5}^2 \simeq \frac{Q_{1,5}^{\text{bps}}}{M} - \frac{1}{2}$, $c_p^2 = \frac{Q_p^{\text{bps}}}{M} + \frac{1}{2} + \frac{\varepsilon}{M}$, and $s_p^2 = \frac{Q_p^{\text{bps}}}{M} - \frac{1}{2} + \frac{\varepsilon}{M}$. The expansion of $a_{1,2}$ in the small M regime then gives

$$a_{1,2} = -(J_\phi + J_\psi) \frac{\sqrt{\gamma}}{\eta} \frac{1}{\sqrt{M}} \mp \frac{1}{4} (J_\phi - J_\psi) \frac{\sqrt{M}}{\sqrt{\gamma}} + \mathcal{O}(M^{3/2}), \quad (\text{A.23})$$

where we have defined $\eta \equiv Q_1^{\text{bps}} Q_5^{\text{bps}} + Q_1^{\text{bps}} Q_p^{\text{bps}} + Q_5^{\text{bps}} Q_p^{\text{bps}} + (Q_1^{\text{bps}} + Q_5^{\text{bps}}) \varepsilon$, and $\gamma \equiv Q_1^{\text{bps}} Q_5^{\text{bps}} Q_p^{\text{bps}} + Q_1^{\text{bps}} Q_5^{\text{bps}} \varepsilon$. Now, one must take appropriate limits of a_1 and a_2 such that they keep finite and the angular momenta is kept fixed. But in (A.23) one sees that, for non-vanishing charges $Q_i^{\text{bps}} \neq 0$ ($i = 1, 5, p$), $a_{1,2}$ diverge as $1/\sqrt{M}$ when we take $M \rightarrow 0$. We can avoid this divergence by imposing that $J_\phi + J_\psi \rightarrow 0$ in the near-BPS limit. Note that as a consequence, in the limit $\varepsilon \rightarrow 0$, the BPS solution must have angular momenta satisfying the relation (3.13)²¹. Under this condition, we can now take a small ε expansion in (A.23) and get

$$a_{1,2} = \pm \sqrt{M} \frac{J_\phi^{\text{bps}}}{2\sqrt{Q_1^{\text{bps}} Q_5^{\text{bps}} Q_p^{\text{bps}}}} [1 + \mathcal{O}(\varepsilon)]. \quad (\text{A.24})$$

Use of (A.20) and (A.24) in (3.9), (3.12) and (3.11) yields straightforwardly the near-BPS expansions for the temperature, (3.15), for the entropy, (3.19), and for the angular velocities, (3.16), respectively.

The off-BPS expansion of the electric potential $\Phi^{(p)}$ leading to (3.16) is straightforward. However, the expansion of the D1 and D5 electric potentials is more subtle. Indeed, if in (3.11) we do the most natural step, $(\tanh \delta_1) c_1 c_5 c_p - (\coth \delta_1) s_1 s_5 s_p = s_1 c_5 c_p - c_1 s_5 s_p$ we just catch the BPS value but not the next order term of the expansion. To capture the next order off-BPS contribution one has to introduce the parameter M that measures the energy above extremality. This is consistently done with the following step: $(\tanh \delta_1) c_1 c_5 c_p = c_1 c_5 c_p \frac{M s_1 c_1}{M(1+s_1^2)}$ (and similarly for the term proportional to $\coth \delta_1$). Then, use of $M s_1^2 \simeq Q_1^{\text{bps}} - M/2$ and $M = 4\sqrt{Q_p^{\text{bps}}} \sqrt{\varepsilon}$ allows to finally write $(\tanh \delta_1) c_1 c_5 c_p \simeq c_1 c_5 c_p (1 - q\sqrt{\varepsilon})$, where q is a ratio of BPS charges. The expansion (3.16) for $\Phi^{(1)}$, $\Phi^{(5)}$ now follows naturally.

B Explicit agreement for other black hole systems

In this Appendix we will perform the Euclidean zero temperature limit and study the statistical mechanics of some BHs that have not been considered in the main body of the text. The main

²¹Alternatively, note that we could relax this condition in the off-BPS regime. That is we could instead fix J_ϕ and let J_ψ arbitrary “during” the near-BPS approach, as long as in the BPS limit one ended with $J_\phi + J_\psi = 0$. Our final result is independent of the particular off-BPS path choosen.

motivation to do this is two-folded. First, we explicitly verify that the relation between the Euclidean zero temperature and Sen's entropy formalisms is indeed general and not restricted to the three-charged BH studied in the main body of the text. Second, we get a list of conjugated chemical potentials for several BH systems. With these at hand we can also study the thermodynamics of the dual CFT. We consider some relevant asymptotically flat systems that have been discussed within Sen's formalism context in [9], namely: the four-charged BH (subsection B.1), and the Kerr-Newman BH (subsection B.2). The agreement between the two formalisms is also confirmed for black holes of gauged supergravity elsewhere [12, 15].

B.1 Four-charged black holes

We study the statistical properties at zero temperature of the asymptotically flat four-charged BH in four dimensions ($4D$). This system has three distinct extreme cases: the BPS BH (studied in subsection B.1.1), the ergo-free branch family of BHs (subsection B.1.2), and the ergo-branch family (subsection B.1.3). These last two are extreme but not BPS BHs and we are following the nomenclature of [9].

The most general non-extremal rotating four-charged BH was first found in [37] as a solution of heterotic string theory compactified on a six-torus. The four gauge fields of the solution were however not explicitly given. This BH is also a solution of $N = 2$ supergravity coupled to three vector multiplets, which in turn can be consistently embedded in $N = 8$ maximal supergravity [37, 36, 38]. As first observed for the static non-extreme case [36], these theories can also be obtained from compactification of type II supergravity on $T^4 \times S^1 \times \tilde{S}^1$. Therefore, from the $10D$ viewpoint these BHs have a D-brane interpretation, *e.g.*, they describe the D2-D6-NS5-P solution of type IIA supergravity or the D1-D5-KK-P solution of type IIB supergravity (or any dual system to these obtained by U -dualities).

Take $N = 2$ supergravity coupled to three vector multiplets. The field content of the theory is: the graviton $g_{\mu\nu}$, four gauge fields $A_{(1)1,2}, \hat{\mathcal{A}}_{(1)}^{1,2}$, three dilatons φ_i and three axions χ_i (with $1 \leq i \leq 3$). The full solution can be explicitly found in [38]. Compared with [38], we use the parameters $\mu \equiv 4m$ and $l \equiv 4a$ that avoid nasty factors of 4 in the thermodynamic quantities. The horizons of the solution are at

$$r_{\pm} = \frac{1}{4} \left(\mu \pm \sqrt{\mu^2 - l^2} \right), \quad (\text{B.1})$$

and thus the system has regular horizons when $\mu \geq |l|$. When $l = 0$ we recover the static solutions found in [36].

The conserved mass E , angular momentum J , and gauge charges Q_i 's of the BH are (we use $G_4 \equiv 1/8$ for this system)

$$\begin{aligned} E &= \frac{\mu}{2} \sum_{i=1}^4 \cosh(2\delta_i), & J_{\phi} &= \frac{1}{2} \mu l (c_1 c_2 c_3 c_4 - s_1 s_2 s_3 s_4), \\ Q_i &= \mu s_i c_i, & i &= 1, 2, 3, 4, \end{aligned} \quad (\text{B.2})$$

which are invariant under interchange of the δ_i 's, as expected from the U -duality relations.

The left and right movers inverse temperatures, the entropy, electric potentials and angular

velocity are [39],

$$\begin{aligned}
\beta_L &= 2\pi\mu (c_1 c_2 c_3 c_4 - s_1 s_2 s_3 s_4) , & \beta_R &= \frac{2\pi\mu^2}{\sqrt{\mu^2 - l^2}} (c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4) , \\
S &= \pi\mu^2 (c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4) + \pi\mu\sqrt{\mu^2 - l^2} (c_1 c_2 c_3 c_4 - s_1 s_2 s_3 s_4) , \\
\Phi^{(i)} &= \frac{\pi\mu}{\beta} [(\tanh \delta_i) c_1 c_2 c_3 c_4 - (\coth \delta_i) s_1 s_2 s_3 s_4] \\
&\quad + \frac{\pi\mu^2}{\beta\sqrt{\mu^2 - l^2}} [(\tanh \delta_i) c_1 c_2 c_3 c_4 + (\coth \delta_i) s_1 s_2 s_3 s_4] , \quad i = 1, 2, 3, 4 , \\
\Omega &= \frac{1}{\beta} \frac{2\pi l}{\sqrt{\mu^2 - l^2}} .
\end{aligned} \tag{B.3}$$

B.1.1 BPS black hole

The BPS limit of the four charged BH is obtained by taking $\mu \rightarrow 0$, $\delta_i \rightarrow \infty$, while keeping Q_i fixed ($i = 1, 2, 3, 4$), and $l \rightarrow 0$ at the same rate as μ , *i.e.*, $l/\mu \rightarrow 1$. As a consequence $J \rightarrow 0$ and the BPS four-charged BH is non-rotating²². Therefore, the BPS charges satisfy the BPS constraints,

$$E^{bps} = Q_1^{bps} + Q_2^{bps} + Q_3^{bps} + Q_4^{bps} , \quad J^{bps} = 0 , \tag{B.4}$$

where $Q_i^{bps} = \mu e^{2\delta_i}/4$. To study the thermodynamics near the $T = 0$ BPS solution we work in the *near*-BPS limit. We take

$$\mu \rightarrow 0 , \quad \delta_{1,2,3} \rightarrow \infty , \quad Q_{1,2,3} \text{ fixed} ; \quad \delta_4 \text{ finite} ; \quad l \rightarrow 0 \quad (l/\mu \rightarrow 1) . \tag{B.5}$$

Note that we take the four boosts to be positive and we choose to keep δ_4 finite, without any loss of generality (due to U -dualities).

Define the off-BPS parameter above extremality ε , to be $\varepsilon = \mu e^{-2\delta_4}/4$ so that $E \equiv E^{bps} + \varepsilon$. The procedure yielding the off-BPS expansion of the several thermodynamic quantities is quite similar to the one done in the three-charged BH (see Appendix A.2). So we just quote the relevant results.

Expanding the left and right temperatures in terms ε yields,

$$\beta_L = \pi \sqrt{\frac{Q_1^{bps} Q_2^{bps} Q_3^{bps}}{Q_4^{bps}}} , \quad \beta_R = \pi \sqrt{Q_1^{bps} Q_2^{bps} Q_3^{bps}} \frac{1}{\sqrt{\varepsilon}} . \tag{B.6}$$

The BPS limit corresponds to send $\beta_R \rightarrow \infty$, and we now can use β_R as the off-BPS parameter, instead of ε .

The expansion in β_R of the conserved charges is

$$\begin{aligned}
E &= E^{bps} + \mathcal{O}(\beta_R^{-2}) , & J &= \frac{\pi Q_1^{bps} Q_2^{bps} Q_3^{bps}}{\beta_R} + \mathcal{O}(\beta_R^{-2}) , \\
Q_{1,2,3} &\simeq Q_{1,2,3}^{bps} , & Q_4 &= Q_4^{bps} + \mathcal{O}(\beta_R^{-2}) .
\end{aligned} \tag{B.7}$$

²²The reason being that the roots that define the horizon are (B.1), and thus $\mu \geq |l|$ must hold to have a regular solution.

The remaining thermodynamic quantities have the expansion,

$$\begin{aligned} S &= S^{bps} + \mathcal{O}(\beta_R^{-1}), \quad \Omega = \frac{4\pi}{\beta_R} + \mathcal{O}(\beta_R^{-2}), \\ \Phi^{(i)} &= \Phi_{bps}^{(i)} - \frac{2\phi_i}{\beta_R} + \mathcal{O}(\beta_R^{-2}), \quad i = 1, 2, 3, 4, \end{aligned} \quad (\text{B.8})$$

where

$$\begin{aligned} S^{bps} &= 2\pi \left[Q_1^{bps} Q_2^{bps} Q_3^{bps} Q_4^{bps} \right]^{1/2}, \\ \Phi_{bps}^{(i)} &= 1, \quad \phi_i = \frac{\pi \left[Q_1^{bps} Q_2^{bps} Q_3^{bps} Q_4^{bps} \right]^{1/2}}{Q_i^{bps}}, \quad i = 1, 2, 3, 4. \end{aligned} \quad (\text{B.9})$$

The last relation gives the key quantities, namely the conjugated potentials ϕ_i 's of the solution that have an important role in the dual CFT. The expressions of the BPS entropy S_{bps} , and conjugated potentials ϕ_i 's agree with the corresponding quantities computed in [9] using Sen's entropy function formalism²³.

The SQSR for the four-charge BH is then

$$I_{bps} = \phi_1 Q_1^{bps} + \phi_2 Q_2^{bps} + \phi_3 Q_3^{bps} + \phi_4 Q_4^{bps} - S_{bps}. \quad (\text{B.10})$$

B.1.2 Extreme (non-BPS) black hole: ergo-free solution

In the four-charged system we can take an extremal limit that yields a rotating BH without ergosphere. For this reason, this BH was dubbed ergo-free solution in [9].

This limit is similar to the BPS regime taken in the previous Appendix B.1.1 in which we send the boosts to infinity; the difference being that we take an odd number (one, for definiteness, but it could as well be three) of boosts to be negative. As explained in a similar context in section 5, this limit yields an extreme, but not BPS, BH.

Concretely, take the *near*-extremal limit ($\delta_{1,2,3} > 0$; $\delta_4 < 0$, $Q_4 < 0$):

$$\mu \rightarrow 0, \quad \delta_{1,2,3} \rightarrow \infty, \quad Q_{1,2,3} \text{ fixed}; \quad \delta_4 < 0 \text{ finite}; \quad \frac{l}{\mu} \rightarrow \frac{J}{\sqrt{-Q_1 Q_2 Q_3 Q_4}}. \quad (\text{B.11})$$

The charges in the extreme solution satisfy the constraint

$$E^{ext} = Q_1^{ext} + Q_2^{ext} + Q_3^{ext} - Q_4^{ext}, \quad (\text{B.12})$$

where $Q_{1,2,3}^{bps} = \mu e^{2\delta_{1,2,3}}/4$, $Q_4^{ext} = -\mu e^{-2\delta_4}/4$, and J^{ext} is arbitrary. Using the off-extremality parameter, $\varepsilon = \mu e^{2\delta_4}/4 = \pi^2 Q_1^{ext} Q_2^{ext} Q_3^{ext} \beta_L^{-2}$ (so the extremal limit is obtained by sending $\beta_L \rightarrow \infty$), we get the following expansion for the relevant thermodynamic quantities:

$$\begin{aligned} S &= S_{ext} + \mathcal{O}(\beta_L^{-1}), \quad \Omega = \Omega_{ext} - \frac{2\omega}{\beta_L} + \mathcal{O}(\beta_L^{-2}), \\ \Phi^{(i)} &= \Phi_{ext}^{(i)} - \frac{2\phi_i}{\beta_L} + \mathcal{O}(\beta_L^{-2}), \quad i = 1, 2, 3, 4, \end{aligned} \quad (\text{B.13})$$

²³Once we match the notation $\omega \equiv 2\pi\alpha$ and $\phi_i \equiv 2\pi e_i$ and we take into consideration that we use $G_4 \equiv 1/8$, while [9] uses $G_4 \equiv 1/(16\pi)$.

where

$$\begin{aligned} S_{ext} &= 2\pi \left[-Q_1^{ext} Q_2^{ext} Q_3^{ext} Q_4^{ext} - (J^{ext})^2 \right]^{1/2}, \\ \Omega_{ext} &= 0, \quad \Phi_{ext}^{(1,2,3)} = 1, \quad \Phi_{ext}^{(4)} = -1. \end{aligned} \quad (\text{B.14})$$

The conjugated potentials are

$$\begin{aligned} \omega &= - \frac{2\pi J^{ext}}{\left[-Q_1^{ext} Q_2^{ext} Q_3^{ext} Q_4^{ext} - (J^{ext})^2 \right]^{1/2}} \\ \phi_i &= - \frac{\pi Q_1^{ext} Q_2^{ext} Q_3^{ext} Q_4^{ext}}{Q_i^{ext} \left[-Q_1^{ext} Q_2^{ext} Q_3^{ext} Q_4^{ext} - (J^{ext})^2 \right]^{1/2}}, \quad i = 1, 2, 3, 4. \end{aligned} \quad (\text{B.15})$$

Again, these expressions for S_{ext} , ω and ϕ_i 's match the ones found in [9] using Sen's entropy function formalism (see footnote 23 for normalization conventions).

Although this is a non-BPS solution, it satisfies the extremal constraint (B.12) that is linear in the charges. Using in addition (B.14), we find that (5.3), applied to this system, vanishes and the QSR for this system simplifies to

$$I_{ext} = \sum_{i=1}^4 \phi_i Q_i^{ext} + \omega J^{ext} - S_{ext}. \quad (\text{B.16})$$

This is an example of a rotating extreme solution without ergosphere. It has a finite on-shell action.

B.1.3 Extreme (non-BPS) black hole: ergo-branch solution

This time we take the limit $\mu \rightarrow l$. This yields an extreme BH with an ergosphere that was coined as ergo-branch solution in [9] (This is the four-charged counterpart of the solution studied in Section 5.1).

We take the near-extreme limit

$$\mu \rightarrow l + \varepsilon, \quad \varepsilon \ll 1. \quad (\text{B.17})$$

When the off-extreme parameter ε vanishes, the temperature indeed vanishes since $\beta_R \rightarrow \infty$ in (B.3). The off-extreme expansion of the conserved charges (B.2) around the corresponding extreme values (obtained by replacing μ by l in (B.2)) is straightforward, and the expansion of the thermodynamic quantities (B.3) yields²⁴

$$\begin{aligned} \beta_L &= 2\pi l (c_1 c_2 c_3 c_4 - s_1 s_2 s_3 s_4) + \mathcal{O}(\varepsilon), \quad \beta_R = \sqrt{2}\pi l^{3/2} (c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4) \frac{1}{\sqrt{\varepsilon}}, \\ S &= S_{ext} + \mathcal{O}(\beta_R^{-1}), \quad \Omega = \Omega_{ext} - \frac{2\omega}{\beta_R} + \mathcal{O}(\beta_R^{-2}), \\ \Phi^{(i)} &= \Phi_{ext}^{(i)} - \frac{2\phi_i}{\beta_R} + \mathcal{O}(\beta_R^{-2}), \quad i = 1, 2, 3, 4, \end{aligned} \quad (\text{B.18})$$

²⁴We use the relation $\sqrt{\mu^2 - l^2} \simeq 2\pi\mu^2 (c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4) / \beta_R$

where the extreme values satisfy

$$\begin{aligned} S_{ext} &= 2\pi [Q_1^{ext} Q_2^{ext} Q_3^{ext} Q_4^{ext} + (J^{ext})^2]^{1/2}, & \Omega_{ext} &= 2l^{-1} (c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4)^{-1}, \\ \Phi_{ext}^{(i)} &= \frac{(\tanh \delta_i) c_1 c_2 c_3 c_4 + (\coth \delta_i) s_1 s_2 s_3 s_4}{c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4}, & i &= 1, 2, 3, 4, \end{aligned} \quad (\text{B.19})$$

and the conjugated potentials are

$$\begin{aligned} \omega &= \frac{2\pi J^{ext}}{[Q_1^{ext} Q_2^{ext} Q_3^{ext} Q_4^{ext} + (J^{ext})^2]^{1/2}}, \\ \phi_i &= \frac{2\pi l^2}{Q_i^{ext}} \frac{s_1 c_1 s_2 c_2 s_3 c_3 s_4 c_4}{c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4}, & i &= 1, 2, 3, 4. \end{aligned} \quad (\text{B.20})$$

Note that in the last expression could be rewritten only in terms of the conserved charges as expected by the attractor mechanism. We do not do it here because the expression is too long. The expressions of the extremal entropy S_{ext} , and conjugated potentials ω and ϕ_i 's agree with the corresponding quantities computed in [9] using Sen's entropy function formalism (see footnote 23 for normalization conventions).

The QSR for this system is

$$I = \beta \left(E^{ext} - \sum_{i=1}^4 \Phi_{ext} Q_i^{ext} - \Omega_{ext} J^{ext} \right) + \sum_{i=1}^4 \phi_i Q_i^{ext} + \omega J^{ext} - S_{ext} + \mathcal{O}(\beta_R^{-1}) \quad (\text{B.21})$$

In the supersymmetric system the analogue of the first term vanishes due to the BPS constraint on the conserved charges. But, in general, for non-BPS extreme BHs it does not vanish (see also discussion associated with (5.3)). In the present case the factor in between brackets is $-\frac{l}{2} \frac{c_1 c_2 c_3 c_4 - s_1 s_2 s_3 s_4}{c_1 c_2 c_3 c_4 + s_1 s_2 s_3 s_4}$. Note that this quantity vanishes when rotation is absent. When it is present, the solution has an ergosphere and the non-vanishing contribution seems to be associated with its existence, as discussed in section 5.

B.2 Extreme Kerr-Newman black hole

In this section we take the near-extreme limit of the Kerr-Newman BH with ADM mass M , ADM charge Q and ADM angular momentum $J = aM$ that is a solution of the Einstein-Maxwell action $I = \frac{1}{16\pi} \int d^4x \sqrt{-g} (R - F^2)$ (so, we set $G_4 \equiv 1$). In the extreme state the charges satisfy the constraint $M^2 = a^2 + Q^2$, the horizons coincide, $r_{\pm} = M$, and one also has the useful relation $M^2 + a^2 = 2\sqrt{J^2 + Q^4/4}$. Define the off-extremality parameter ε such that $M = M_e + \varepsilon$ which implies that $r_+ \sim M_e + \sqrt{2M_e} \sqrt{\varepsilon}$ (the subscript e stands for the on-shell extreme solution). In terms of the inverse temperature $\beta = \frac{2\pi(r_+^2 + a^2)}{r_+ - M}$ it is given by $\sqrt{\varepsilon} = \frac{2\pi(M_e^2 + a_e^2)}{\sqrt{2M_e}\beta}$. Using the expressions $S = \pi(r_+^2 + a^2)$, $\Omega = a/(r_+^2 + a^2)$ and $\Phi = Qr_+/(r_+^2 + a^2)$ one gets the expansion:

$$\begin{aligned} S &= S_e + \mathcal{O}(\beta^{-1}), & S_e &= 2\pi \sqrt{J_e^2 + Q_e^4/4}; \\ \Omega &= \Omega_e - \frac{\omega}{\beta} + \mathcal{O}(\beta^{-2}), & \Omega_e &= \frac{J_e}{2M_e \sqrt{J_e^2 + Q_e^4/4}}, & \omega &= \frac{2\pi J_e}{\sqrt{J_e^2 + Q_e^4/4}}; \\ \Phi &= \Phi_e - \frac{\phi}{\beta} + \mathcal{O}(\beta^{-2}), & \Phi_e &= \frac{Q_e M_e}{2\sqrt{J_e^2 + Q_e^4/4}}, & \phi &= \frac{\pi Q_e^3}{\sqrt{J_e^2 + Q_e^4/4}}. \end{aligned}$$

The extremal entropy S_e , and conjugated potentials ω and ϕ agree with the corresponding quantities computed in [9] using Sen’s entropy function formalism²⁵.

The QSR for this system is

$$I = \beta (M_e - \Phi_e Q_e - \Omega_e J_e) + \phi Q_e + \omega J_e - S_e + \mathcal{O}(\beta^{-1}) \quad (\text{B.22})$$

The first term does not vanish, a feature that seems to be common to non-BPS extreme black holes with ergosphere. The factor in between brackets is $M_e(M_e^2 - Q_e^2)/(M_e^2 + a_e^2)$. If rotation is absent, $a = 0$, one has $M_e = Q_e$ and the above term vanishes. When it is present, the solution has an ergosphere and the non-vanishing contribution seems to be associated with its existence, as discussed in section 5.

References

- [1] S. Ferrara, R. Kallosh and A. Strominger, “N=2 extremal black holes,” Phys. Rev. D **52** (1995) 5412 [arXiv:hep-th/9508072].
A. Strominger, “Macroscopic Entropy of $N = 2$ Extremal Black Holes,” Phys. Lett. B **383** (1996) 39 [arXiv:hep-th/9602111].
S. Ferrara and R. Kallosh, “Supersymmetry and Attractors,” Phys. Rev. D **54** (1996) 1514 [arXiv:hep-th/9602136].
- [2] G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Corrections to macroscopic supersymmetric black-hole entropy,” Phys. Lett. B **451** (1999) 309 [arXiv:hep-th/9812082].
G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Deviations from the area law for supersymmetric black holes,” Fortsch. Phys. **48** (2000) 49 [arXiv:hep-th/9904005].
G. Lopes Cardoso, B. de Wit and T. Mohaupt, “Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes,” Nucl. Phys. B **567** (2000) 87 [arXiv:hep-th/9906094].
G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, “Stationary BPS solutions in $N = 2$ supergravity with R^{*2} interactions,” JHEP **0012** (2000) 019 [arXiv:hep-th/0009234].
- [3] H. Ooguri, A. Strominger and C. Vafa, “Black hole attractors and the topological string,” Phys. Rev. D **70** (2004) 106007 [arXiv:hep-th/0405146].
- [4] W. Li and A. Strominger, “Supersymmetric probes in a rotating 5D attractor,” [arXiv:hep-th/0605139].
- [5] M. Guica, L. Huang, W. Li and A. Strominger, “ R^{*2} corrections for 5D black holes and rings,” JHEP **0610** (2006) 036 [arXiv:hep-th/0505188].
- [6] J. F. Morales and H. Samtleben, “Entropy function and attractors for AdS black holes,” JHEP **0610** (2006) 074 [arXiv:hep-th/0608044].
- [7] A. Sen, “Black hole entropy function and the attractor mechanism in higher derivative gravity,” JHEP **0509** (2005) 038 [arXiv:hep-th/0506177].

²⁵The match is obtained once the different conventions for the action and G_4 are mapped. In [9], the Maxwell term in the action has an extra factor of $1/4$ and $G_4 \equiv 1/(16\pi)$.

- [8] A. Sen, “Entropy function for heterotic black holes,” JHEP **0603** (2006) 008 [arXiv:hep-th/0508042].
- [9] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, “Rotating attractors,” JHEP **0610** (2006) 058 [arXiv:hep-th/0606244].
- [10] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, “An index for 4 dimensional super conformal theories,” [arXiv:hep-th/0510251].
- [11] M. Berkooz, D. Reichmann and J. Simon, “A Fermi surface model for large supersymmetric AdS(5) black holes,” JHEP **0701**, 048 (2007) [arXiv:hep-th/0604023].
- [12] P. J. Silva, “Thermodynamics at the BPS bound for black holes in AdS,” JHEP **0610**, 022 (2006) [arXiv:hep-th/0607056].
- [13] P. J. Silva, “Phase transitions and statistical mechanics for BPS black holes in AdS/CFT,” JHEP **0703**, 015 (2007) [arXiv:hep-th/0610163].
- [14] G. W. Gibbons, M. J. Perry and C. N. Pope, “The first law of thermodynamics for Kerr - anti-de Sitter black holes,” Class. Quant. Grav. **22** (2005) 1503 [arXiv:hep-th/0408217].
- [15] P. J. Silva, “On Uniqueness of supersymmetric Black holes in AdS(5),” arXiv:0712.0132 [hep-th].
- [16] N. V. Suryanarayana and M. C. Wapler, “Charges from Attractors,” [arXiv:hep-th/0704.0955].
- [17] A. Ghodsi, “R**4 corrections to D1D5p black hole entropy from entropy function formalism,” Phys. Rev. D **74**, 124026 (2006) [arXiv:hep-th/0604106].
- [18] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest and A. Van Proeyen, “New formulations of D = 10 supersymmetry and D8 - O8 domain walls,” Class. Quant. Grav. **18** (2001) 3359 [arXiv:hep-th/0103233].
- [19] S. Corley, A. Jevicki and S. Ramgoolam, “Exact correlators of giant gravitons from dual N = 4 SYM theory,” Adv. Theor. Math. Phys. **5** (2002) 809 [arXiv:hep-th/0111222].
- [20] D. Berenstein, “A toy model for the AdS/CFT correspondence,” JHEP **0407**, 018 (2004) [arXiv:hep-th/0403110].
- [21] M. M. Caldarelli and P. J. Silva, “Giant gravitons in AdS/CFT. I: Matrix model and back reaction,” JHEP **0408** (2004) 029 [arXiv:hep-th/0406096].
- [22] M. Cvetič and D. Youm, “General Rotating Five Dimensional Black Holes of Toroidally Compactified Heterotic String,” Nucl. Phys. B **476** (1996) 118 [arXiv:hep-th/9603100].
- [23] S. Giusto, S. D. Mathur, and A. Saxena, “Dual geometries for a set of 3-charge microstates,” Nucl. Phys. B **701**, 357 (2004) [arXiv:hep-th/0405017].
- [24] M. Cvetič and F. Larsen, “Near horizon geometry of rotating black holes in five dimensions,” Nucl. Phys. B **531** (1998) 239 [arXiv:hep-th/9805097].

- [25] A. Dabholkar, A. Sen and S. P. Trivedi, “Black hole microstates and attractor without supersymmetry,” JHEP **0701** (2007) 096 [arXiv:hep-th/0611143].
- [26] D. Astefanesei, K. Goldstein and S. Mahapatra, “Moduli and (un)attractor black hole thermodynamics,” [arXiv:hep-th/0611140].
- [27] R. M. Wald, “Black hole entropy in the Noether charge,” Phys. Rev. D **48** (1993) 3427 [arXiv:gr-qc/9307038].
T. Jacobson, G. Kang and R. C. Myers, “On Black Hole Entropy,” Phys. Rev. D **49** (1994) 6587 [arXiv:gr-qc/9312023].
V. Iyer and R. M. Wald, “Some properties of Noether charge and a proposal for dynamical black hole entropy,” Phys. Rev. D **50** (1994) 846 [arXiv:gr-qc/9403028].
- [28] H. W. Braden, J. D. Brown, B. F. Whiting and J. W. York, “Charged black hole in a grand canonical ensemble,” Phys. Rev. D **42** (1990) 3376.
- [29] R. Emparan and G. T. Horowitz, “Microstates of a neutral black hole in M theory,” Phys. Rev. Lett. **97** (2006) 141601 [arXiv:hep-th/0607023].
R. Emparan and A. Maccarrone, “Statistical description of rotating Kaluza-Klein black holes,” [arXiv:hep-th/0701150].
- [30] T. Jacobson and R. C. Myers, ‘Black Hole Entropy And Higher Curvature Interactions,’ Phys. Rev. Lett. **70** (1993) 3684 [arXiv:hep-th/9305016].
W. Nelson, “A Comment on black hole entropy in string theory,” Phys. Rev. D **50** (1994) 7400 [arXiv:hep-th/9406011].
V. Iyer and R. M. Wald, “A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes,” Phys. Rev. D **52** (1995) 4430 [arXiv:gr-qc/9503052].
- [31] V. Jejjala, O. Madden, S. F. Ross and G. Titchener, “Non-supersymmetric smooth geometries and D1-D5-P bound states,” Phys. Rev. D **71** (2005) 124030 [arXiv:hep-th/0504181].
- [32] J. C. Breckenridge, D. A. Lowe, R. C. Myers, A. W. Peet, A. Strominger and C. Vafa, “Macroscopic and Microscopic Entropy of Near-Extremal Spinning Black Holes,” Phys. Lett. B **381** (1996) 423 [arXiv:hep-th/9603078].
- [33] A.A. Tseytlin, “Extreme dyonic black holes in string theory,” Mod. Phys. Lett. A **11**, 689 (1996) [arXiv:hep-th/9601177];
J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, “D-branes and spinning black holes,” Phys. Lett. B **391**, 93 (1997) [arXiv:hep-th/9602065].
- [34] V. Cardoso, O. J. C. Dias, J. L. Hovdebo and R. C. Myers, “Instability of non-supersymmetric smooth geometries,” Phys. Rev. D **73** (2006) 064031 [arXiv:hep-th/0512277].
- [35] G. W. Gibbons, H. Lu, D. N. Page and C. N. Pope, “The general Kerr-de Sitter metrics in all dimensions,” J. Geom. Phys. **53** (2005) 49 [arXiv:hep-th/0404008].
- [36] G. T. Horowitz, D. A. Lowe and J. M. Maldacena, “Statistical Entropy of Nonextremal Four-Dimensional Black Holes and U-Duality,” Phys. Rev. Lett. **77** (1996) 430 [arXiv:hep-th/9603195].

- [37] M. Cvetič and D. Youm, “Entropy of Non-Extreme Charged Rotating Black Holes in String Theory,” *Phys. Rev. D* **54** (1996) 2612 [arXiv:hep-th/9603147].
- [38] Z. W. Chong, M. Cvetič, H. Lu and C. N. Pope, “Charged rotating black holes in four-dimensional gauged and ungauged supergravities,” *Nucl. Phys. B* **717** (2005) 246 [arXiv:hep-th/0411045].
- [39] M. Cvetič and F. Larsen, “Greybody factors for rotating black holes in four dimensions,” *Nucl. Phys. B* **506** (1997) 107 [arXiv:hep-th/9706071].